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Computing steady incompressible flows past blunt bodies—A historical overview

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1 Introduction

There is a special fascination associated with problems which

1. are very easy to pose
2. the final answer is unexpected, but still has a very clear form
3. the process to arrive at this answer is lengthy and requires input from diverse sources (and the solution process has helped to advance the investigative tools employed).

Although there may be many more surprises left, and several technical details in the present models are extremely complex, the problem of determining the structure of steady incompressible flows past 'simple' blunt bodies (like a cylinder, a sphere, a flat plate perpendicular to free stream, arrays of such bodies etc.) can be said to fall in this category.

In spite of the fact that experiments become time-dependent at relatively low Reynolds numbers (due to instabilities), there are several reasons for studying steady (unstable) flow fields at high Re:

1. To better understand how fluids can 'behave', it is important to characterise a selection of generic flow situations (for example, the form of steady wakes was not theoretically anticipated, but it connects together many older theoretical results).
2. A powerful technique to find (and then possibly exploit) novel flow regimes is to first determine steady solutions with methods immune to instabilities and then separately analyze their (in-)stabilities. As an example flow past a sphere may be possible to stabilise with flow control methods—for reductions in drag of one to two orders of magnitude.
3. Each of the three major investigative techniques—experiment, theory and computer simulation—have been applied in many attempts to reach high Reynolds numbers. For the latter two approaches in particular, the flow past blunt bodies has served as a sufficiently challenging problem to promote significant advances in the techniques themselves.

4. From a numerical perspective, the problem is especially interesting because the insights gained from computing have not been merely quantitative, but highly qualitative as well. With no need to impose any preconceived notions on the form of the solution (as would commonly be needed with, say, asymptotic analysis), unexpected phenomena can be found.

An in-depth study of this subject is far beyond the scope of this brief presentation. In very broad terms, we will outline a few past studies in order to illustrate how experiment, theory and computations—helping each other along—have led to our current understanding of steady high-speed flow solutions to the incompressible Navier-Stokes equations. The focus will primarily be on computations. Some key theoretical and experimental studies will be mentioned briefly.

During the last six decades that numerical techniques have been applied to this problem, the computational resources available to the different investigators have increased from several years on a mechanical hand calculator to several hundred hours on a Cray supercomputer—approximately a factor of 10^8. The numerical algorithms themselves have also improved dramatically. However the corresponding gain in Reynolds number (Re) has only been a factor of about 20. Although limited, it is enough for us to reach a flow regime where a combination of some further asymptotics and numerics might resolve the structure of the flows for all Re. (On the other hand, this was also believed—erroneously—50 years ago...)

2 Background

Chart 1 schematically illustrates steady flows at low Reynolds numbers (a non- dimensional quantity Re=Ud/ν where U is the free stream viscosity, d is the body diameter and ν the kinematic velocity). In the case of a cylinder, the first instability (for increasing Re) is a 'Hopf bifurcation'—the wake starts to oscillate across the centreline and ultimately begins shedding a row of vortices downstream. In the case of a sphere, both the critical Re and the nature of the first instability are less clear (cf. Taneda 1956, Nakamura 1976). For the reasons outlined in the introduction, we are interested in the steady (but unstable) solutions at still higher values of Re for these problems.

Chart 2 shows the governing equations for 2-D in the streamfunction-vorticity form. This form is convenient also for axi-symmetric 3-D problems but not practical in general 3-D. Although Newton (1687) and Daniel and Johann Bernoulli (both in 1738) discussed laws for fluid motion, the first complete description was obtained by Euler in 1755, for inviscid flow. Subsequently, the viscous terms were obtained independently by Navier (1822) and Stokes (1845).

Several closed form solutions are known to both Navier-Stokes (NS) and Euler equations. Those listed at the top of Chart 3 do not describe any common flow situations (but are still sometimes useful in testing NS solvers). Kovasznay (1948) suggested that his solution might describe the closure of wakes for flow past a periodic array of bodies (although this now appears unlikely). A similar solution, possibly relevant to the closure of wide wakes (Peregrine, 1985), was given by Jeffery (1915).

In 2-D Euler solutions for irrotational flows ($\omega = 0$), the streamfunction satisfies Laplace's equation. The invariance of this equation under conformal mappings provides a simple method to generate Euler flows past many bodies, e.g. the attached flow past a cylinder (top left, Chart 4). Although a solution for Re=∞, this is very unlikely to represent a limit for Re$\rightarrow \infty$. If it were, the boundary layer equations for the front surface would develop a singularity around 110° from the stagnation point, an event typically associated with a boundary layer separation. Helmholtz (1868) introduced the idea of vortex sheets—slip planes in the fluid—and found that, using them as additional 'building blocks', more realistic Euler solutions could be obtained. An example of this is the flow past a flat plate due to Kirchhoff, 1869.

Another class of Euler solutions is shown in the lower half of Chart 4. These will prove to be of particular interest in modelling wakes. It is easy to see that two point vortices—of equal strength but with opposite sign—will induce matching velocities on each other, and the pair will translate with constant velocity through ideal fluid. Instead of considering infinite vorticity spread over zero area, we can find translating solutions with piecewise constant vorticity spread over finite areas. If the areas are small, their shapes will be nearly circular. The extreme case—that of the two areas meeting along a stretch of the centreline—has one free parameter. Changing this parameter will affect the translation speed, level of vorticity and size of the vortex pair—but will leave the shape invariant. In particular, the aspect ratio length/width is a constant, approximately 1.6691. Curiously, the equivalent 3-D axi-symmetric solution is much simpler; the Hill's spherical vortex (Hill, 1894) is indeed perfectly spherical, has ω inside it proportional to the distance to the symmetry axis and outside it $\omega = 0$. Vortex sheets can be added at the edges of these translating solutions to provide one additional free parameter; such solutions in 2-D were described (and calculated) by Sadovskii (1971).

3 Some developments on the flow problem

Chart 5 gives a time line of some notable contributions to the problem. If the experiments look under-represented here, we should bear in mind that one of the most intriguing aspects of this problem is just how inaccessible it is to experimental investigations.

The first numerical contribution quoted here, Richardson's pioneering paper of 1910, does not specifically address flow problems. I think, however, it is here that a numerical 'story' should start. Two quotes from Richardson's introduction illustrate how new the concept of finite differences was at this time:

> *Step-by-step arithmetical methods of solving ordinary difference equations have long been employed for the calculation of interest and annuities. Recently their application to differential equations has been very greatly improved by the introduction of rules allied to those for approximate quadrature.*
>
> *The extension to three variables is, however, perfectly obvious. One has only to let the third variable be represented by the number of the page of a book of tracing paper.*

As a first example of finite difference (FD) schemes, Richardson chose the heat equation (Chart 6), proposed an explicit 'leap-frog-type' scheme and calculated the solution for five time steps (using a rather inaccurate approximation to the analytical solution on the second time level to get started). As late as 1937, Richardson's scheme for the heat equation was referred to without any fundamental flaws being recognised (Hartree et al.; the now famous 1928 paper by Courant, Friedrichs and Lewy was not much noted at the time by the numerical community since it discussed FD schemes only as tools for existence proofs, cf. Lax 1967). Richardson was in this 1910 paper close to discovering (and if so, quite certainly also successfully analyzing) the phenomenon of numerical instabilities. The figure below the table on Chart 6 shows the errors at the grid points throughout the first 20 time steps. Had he been able to afford these additional 15 time steps (or had he proceeded just 10 time steps from a more accurate second time level), a key area in numerical analysis could have been advanced by some three decades.

The highlight of the 1910 paper was a FD solution (by a relaxation method) of the elastic equations for a complex geometry. This study was prompted by the recent failure of a dam similar to the (first) dam just built across the Nile at Asswan.

In 1923, Brodetsky generalised Kirchhoff's free streamline solution past a flat plate to the case of a cylinder. The primary difficulty was that the location of the separation point was no longer known. Assuming tangential separation, a unique solution was found with a wake opening up parabolically to downstream infinity. (Neither Kirchhoff nor Brodetsky considered $Re \to \infty$ issues or the possibility of non-uniform vortex sheets)

The next 'landmark' contributions—both numerical and experimental—were by Thom (1927 and 1933, Chart 7). The steady flow for Re=10 was calculated using FD on a rectangular grid (using interpolation at the cylinder surface). During this work a 'different method of procedure presented itself' (Thom, 1933). For the case Re=20, the NS equations were conformally mapped to a simpler domain (on which mesh refinement was used in critical regions), with exceptional accuracy being obtained.

Thom experimentally observed wakes which lengthened until a Karman vortex street began to develop around $Re \approx 40$–60 (sometimes slightly delayed by the channel geometry). The 1964 experimental study by Grove et al reached Re=300 (also in a channel geometry) by placing a horizontal flat plate at the stagnation point at the rear of the wake. The idea of this was to stabilise the flow without otherwise affecting it much. However the longest wake observed (at Re=300) had a length of only L=11, compared to L=40.8 which is now calculated for that case.

A series of theoretical studies, starting with Squire in 1934 (e.g. Imai 1953, Kawaguti 1953, Sychev 1967, 1972, Messiter 1975, Smith 1979, 1983) supported the idea of slender elliptic wakes tending towards Brodetsky's free streamline model (with the internal wake pressure approaching that of the undisturbed flow). The 'triple deck' model in the 1970's represented a major advance in the understanding of the nature of fluid separation at the body. However, a problem concerning the closure of the slender wake was never successfully resolved in these models.

Three studies which disagreed with the long and slender wake concept are particularly noteworthy:

The 1955 study by Allen and Southwell (Chart 8) was carried out at a time when relaxation methods seemed almost limitless in their power and scope. It was very easy at that time to let enthusiasm for numerical results dominate over less exciting error-control issues—especially since the solutions looked very reasonable (and appeared to resolve some key theoretical difficulties). The particular significance of this study from a numerical point of view was the novel method employed for the derivation of quite accurate upwind schemes, a powerful and still often used technique.

A year later (1956), Batchelor proposed a wake model that was very reminiscent of the (erroneous) small wakes calculated by Allen and Southwell. No reference was made to their paper—Batchelor's arguments are strictly theoretical. This study is again one that, although questionable in some key conclusions, contained profound new insights in that it highlighted flaws in the prevalent model. The suggestion for 3-D—a Hill's spherical vortex—is now numerically confirmed (apart from the fact that its size does not appear to remain bounded).

In 1968, Taganoff proposed a model with both length and width of order Re. He then asked Sadovskii to calculate the relevant Euler solutions (cf. the conclusions of the previous chapter and Chart 4). Taganoff's work (available only in Russian) was not noted in the West and was largely ignored in the East (as it was rather heuristic, went against the general views at the time and was followed up with some less good further proposals in subsequent papers).

By the time of my own first computation in this area (1980, Chart 9), the general consensus was strongly in favour of long and slender wakes. The calculation was made possible by some fortunate circumstances:

- During the preceding few years, the direct use of Newton's method on 2-D FD approximations had proven very successful.
- The first vector-supercomputers (CDC Star 100 and Cray 1) were becoming available.
- There was an increasing awareness of the critical role played by far-field boundary conditions (and the need to develop new and accurate approximations that could be used very close to the edge of the wake).

With Newton's method, accurate steady solutions could for the first time be obtained in time-unstable situations (since its quadratic convergence prevents any temporal-type instabilities from developing in the artificial time of the numerical iterations). As a complete surprise, my computations revealed a transition to a much wider wake structure around Re=300. The mechanism for this could be seen clearly, as vorticity started to be convected by the recirculating stream into the wake from its rear. Since this change of trend was unexpected, more error testing and computational refinements were called for. By 1983 (Chart 10), it was clear that the decrease in wake length seen around Re=300 was not correct—but the observation of the transition to wide wakes (and the mechanism behind this) stayed firm.(The excellent $\frac{dL}{dR} = 0.17$ estimate by Smith (1979) was derived for slender wakes—its agreement with the numerics is somewhat surprising).

These wide wakes were subsequently discussed theoretically (in rather general terms) by Peregrine in 1981 and (in more detail) by Smith in 1985. The studies by Chernyshenko (1988; for rows of cylinders in 1992) would appear to be the most successful efforts so far in fitting together a complete picture for this problem. The wake appears to be of the type originally outlined by Taganoff (with vortex sheets being rather insignificant). Very schematically, it consists of a translating Euler-type solution of a size and at a distance from the body adjusted in such a way that:

1. it travels with the same speed as the body;
2. the viscous dissipation of the vorticity in the wake matches the amount generated on the body surface.

Chart 11 shows a similar calculation for a sphere—and the resulting Hill's spherical vortex wake. Although such a vortex is known to be unstable (Pozrikidis, 1986), the instabilities are quite 'gentle' and stabilising flow control might just be possible. Many moving objects (e.g. a golf ball) move at Reynolds numbers close to where the drag suddenly drops. The dimples on its surface induce boundary layer turbulence, thereby delaying the separation and significantly lowering the drag. However, extrapolation

of the calculated drag for steady flows show that the blunt body drag can be lower still, comparable to that for streamlined bodies.

Since the wake becomes wide for flow past a single body, the question of what happens in the case of a row of bodies is particularly intriguing (for references, see the introduction in Ingham et al 1990). With Re and W as free parameters, one might expect singular points in the Re-W-plane. So far, none have been found. Chart 12 shows an example of how the wake changes character when W is increased for a fixed value of Re.

Qualitatively, the wide wakes seem to be rather independent of the shape of the body. Chart 13 shows how flat plates cause much longer (but otherwise similar) wakes than cylinders. This calculation used Newton's method with biquadratic Finite Elements (a combination that proved to be more cost-effective than Newton + 2^{nd} order FD that I had used earlier).

4 Possible future developments, conclusions

Unstructured grids have not yet been used for incompressible flows to the same extent that they have been used for compressible flows. But other major developments—not yet exploited elsewhere—also seem possible. Chart 14 outlines one particularly promising approach allowing a 'black box' code for a time- stepping algorithm to be used to obtain steady flows for values of Re where time stepping alone diverges. Shroff and Keller (1991) further describe how 'arc-length continuation' can be used to follow solution paths through turning- and bifurcation points (as for regular Newton codes).

Even if the flow fields we now see represent the ultimate asymptotic wake structure for steady flows, further challenges remain:

<u>Experiment:</u> - Possible applications for (active) flow control methods.
<u>Theory:</u> - Although the leading terms now seem plausible, many questions remain.
<u>Computation:</u> - Major opportunities in improved algorithms, parallel computing etc.

For all these three investigative techniques, I believe this problem will continue to offer a very fruitful field for further explorations and improvements of methodologies.

5 Acknowledgements

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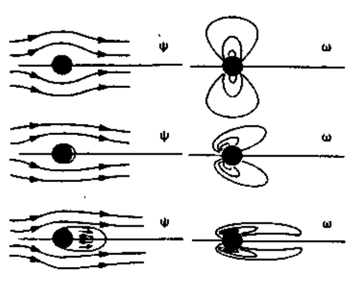
STEADY FLOW FIELDS PAST A CYLINDER AND A SPHERE AT LOW REYNOLDS NUMBERS

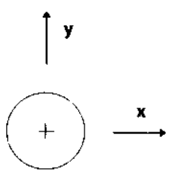


	Critical Reynolds numbers for: CYLINDER	SPHERE
Symmetry up and downwind	0	0
Onset of separation	5	20 ?
Onset of instabilities	40	130 - 210 ?
By passive flow control (channel, splitter plate etc.), steady flows have been obtained up to approximately:	300	500

(Reynolds number based on body diameter)

Chart 1. Schematic illustrations of steady flow fields at low Reynolds numbers.

GOVERNING EQUATIONS — STEADY, INCOMPRESSIBLE FLOW



x – velocity u

y – velocity v

streamfunction ψ satisfies $\frac{\partial\psi}{\partial y} = u\ ,\ \frac{\partial\psi}{\partial x} = -v$

because of incompressibility $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

vorticity $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

Steady NAVIER - STOKES equations (1845):

$$\left\{\begin{array}{l} \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \omega = 0 \\ \frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + \frac{Re}{2}\left\{ \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} \right\} = 0 \end{array}\right.$$

For Re = ∞ , steady EULER equations (1755).

Chart 2. The Navier–Stokes equations in streamfunction–vorticity formulation.

SOME CLOSED FORM SOLUTIONS TO NAVIER-STOKES EQUATIONS

Use Re instead of Re/2 in earlier definition of the NS equations

$\psi = \frac{x^2+y^2}{4}\{2 - \ln(x^2+y^2)\}$ $\omega = \ln(x^2+y^2)$	$\psi = \frac{x-y}{Re} - e^{(x+y)}$ $\omega = 2\ e^{(x+y)}$	$\psi = \frac{2}{Re}\ \text{atan}(y/x) - \left[\frac{x^2+y^2}{4}\right]^2$ $\omega = x^2+y^2$

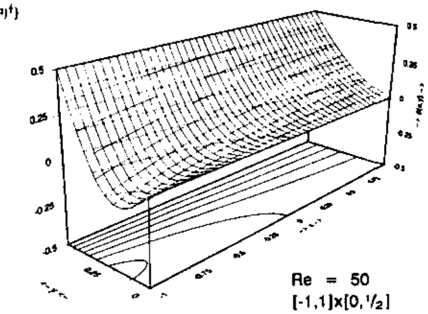
Richards and Crane (1979)

Closure of (some) wake in a channel:

$$\psi = y - \frac{\sin 2\pi y}{2\pi}\ e^{x/2\,\{Re - (Re^2 + 16\pi^2)^{\frac{1}{2}}\}}$$

$$\omega = \frac{Re}{2}\{(Re^2+16\pi^2)^{\frac{1}{2}} - Re\}\,(\psi - y)$$

Kovasznay (1948)



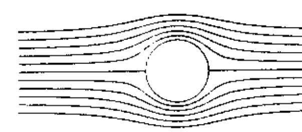
Similar solution in vertically unbounded geometry

Jeffrey (1915), Peregrine (1985)

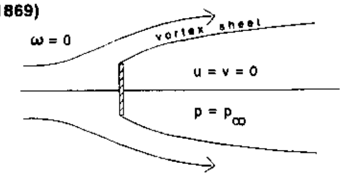
Chart 3. Some closed form solutions to the Navier–Stokes equations.

SOME CLOSED FORM EULER SOLUTIONS IN 2-D

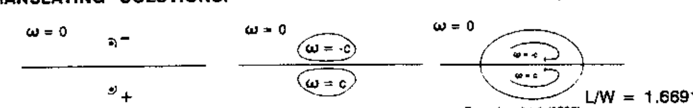
POTENTIAL FLOW PAST A CYLINDER



KIRCHHOFF FREE STREAMLINE FLOW (1869)

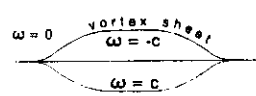


TRANSLATING SOLUTIONS:



Pierrehumbert (1980)
Saffman, Tanveer (1982)
Wu, Overman, Zabusky (1984)

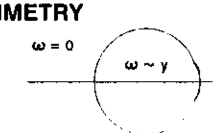
(Moving with uniform speed through the surrounding fluid)



Sadovskii (1971)

3-D AXI-SYMMETRY

Hill's spherical vortex



Hill (1894)

Chart 4. Some closed form solutions to the Euler equations.

STEADY INCOMPREESIBLE FLOW PAST BLUNT BODIES

EXPERIMENT	THEORY	NUMERICAL
	1755 Euler equations	
	1845 Navier-Stokes eq.	
	Free streamline Euler solutions:	
	1868 Jets *Helmholz*	
	1869 Bodies *Kirchhoff*	
	1905 Boundary layers *Prandtl*	
	1910 Linearized NS *Oseen*	**1910 Relaxation methods on applied problems** *Richardson*
	1923 Euler sol. cylinder *Brodetsky*	
1933 *Thom*	**1934 L ~ Re , W ~ Re$^{1/2}$** *Squire*	**1933 FD calc. cyl. Re=20** *Thom*
	1956 L ~ 1 , W ~ 1 *Batchelor*	**1955 FD calc. cyl. up to Re = 1000 (!?)** *Allen, Southwell*
1964 Re = 300 (with splitter plate) *Grove et.al.*	**1965 L ~ Re, W ~ 1** *Acrivos et.al.*	
	Wide wake sugges- tion: L ~ Re , W ~ Re	
1968 *Acrivos et.al.*	**1968** *Taganoff*	
	Developments on model by Squire L ~ Re , W ~ Re$^{1/2}$	**1969 Re = 60** *Takami, Keller*
	1967 1972 *Sychev*	**1970 Re = 100** *Dennis, Chang*
		1973 Spectral + Newton, Re = 40 *Nieuwstadt, Keller*
	1979 1983 *Smith*	**1980 FD+Newton, Re = 300 Wide wakes observed** *BF*
	Wide wake models L ~ Re , W ~ Re	
	1981 *Peregrine*	**1985 Re = 600** *BF*
	1985 *Smith*	**1988 Fl. past sphere, Re = 5000** *BF*
	1988 1992 *Chernyshenko*	**1991 Row of cyl. Re=700, all separations** *BF*
		1991 Hybrid Relax-Newton method *Shroff, Keller*

Chart 5. Some key experimental, theoretical and numerical studies.

1910 Richardson

COMPUTATION

First example: $\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2}$, Init. cond. $\phi(x,0) = 1$, $-\frac{1}{2} < x < \frac{1}{2}$

Bound. cond. $\phi(\pm\frac{1}{2},t) = 0$, $t > 0$

Stencil:

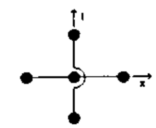
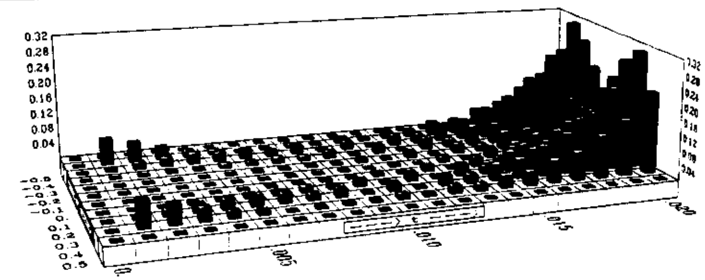


Table: (from original paper)

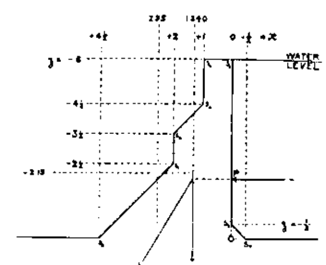
	$t = 0$	0·001.	0·002.	0·003.	0·004.	0·005.	0·005 correct by Fourier's method.	Errors.
0·5	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	
0·4	1·0000	0·5000	0·8356	0·7714	0·7209	0·6729	0·6828	-0·0099
0·3	1·0000	0·9375	0·9434	0·9693	0·9193	0·9729	0·9545	-0·0216
0·2	1·0000	0·9998	0·9993	0·9964	0·9945	0·9887	0·9980	-0·0093
0·1	1·0000	1·0000	1·0000	0·9999	0·9994	0·9990	0·9996	-0·0006
0·0	1·0000	1·0000	1·0000	1·0000	1·0000	0·9997	1·0000	-0·0003
-0·1	1·0000	1·0000	1·0000	0·9999	0·9994	0·9990	0·9996	-0·0006

Errors at different times: (in magnitude)

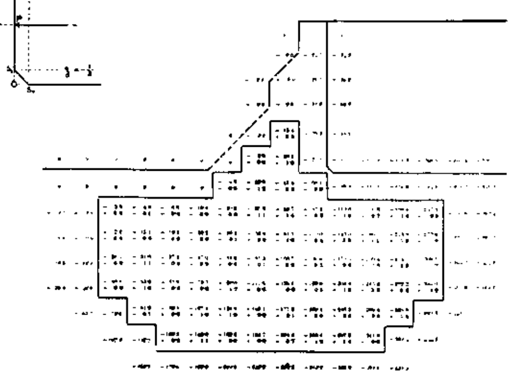


STRESSES IN A MASONRY DAM

CROSS SECTION OF 1902 ASSWAN DAM



STRESS FUNCTION IN FOUNDATION OF DAM



A SERIES OF REFINED CALCULATIONS PER- FORMED ON CRITICAL PARTS OF DAM ITSELF

Chart 6. Richardson (1910)

1927 Thom 1933

COMPUTATION AND EXPERIMENT

Equations:

Streamfunction-vorticity formulation

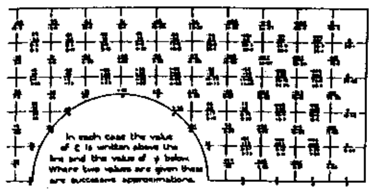
$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega$$

$$\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = -\frac{Re}{2}\left\{\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right\}$$

where

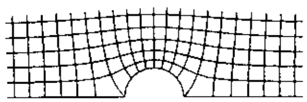
$$u = \frac{\partial\psi}{\partial y},\quad v = -\frac{\partial\psi}{\partial x},\quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Re = 10:



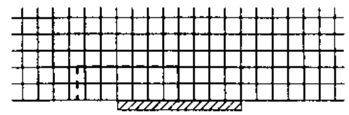
Re = 20: $X = x + iy$, $Z = \xi + i\eta$

Mapping $Z = X + 1/X$



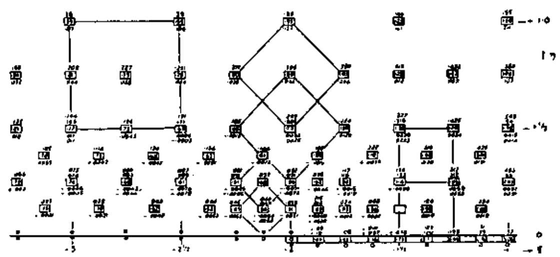
-1 1

X - plane



-2 0 2

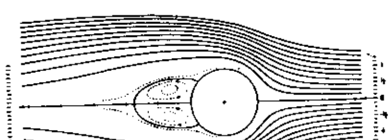
Z - plane



A portion of the working including the eddy region. Figures shown thus 0·089 were first taken from a previous sheet; from these the sloping figures were calculated. Values of ψ are enclosed in rectangles. Values of ζ are written above the rectangle and values of ψ below.

Part of work sheet

Flow field



Calculated streamlines past a circular cylinder at Vd/ν = 20. The spacing of the streamlines in the undisturbed flow is shown by the short lines at each end. The dotted streamlines are at + 1/256, − 1/256, and − 1/128.

$L = 2.76$

Best current value: $L = 2.82$

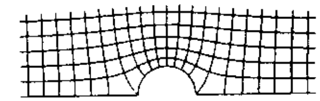
Chart 7. Thom (1927, 1933)

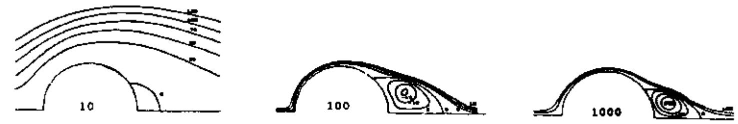
1955 Allen & Southwell

COMPUTATION

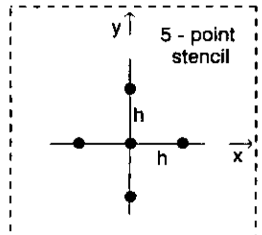
Streamfunction - vorticity formulation

Same mapping as used by Thom





Upwinding - original strategy:



Model eq. for vort. transp-diff in equilibrium: $\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + 2\lambda\frac{\partial\omega}{\partial x} + 2\mu\frac{\partial\omega}{\partial y} = 0$

Separate into two equations:

$$\begin{cases} \frac{\partial^2\omega}{\partial x^2} + 2\lambda\frac{\partial\omega}{\partial x} = A \\ \frac{\partial^2\omega}{\partial y^2} + 2\mu\frac{\partial\omega}{\partial y} = -A \end{cases}$$

Assume λ, μ, A locally constant. Solve each eq. above in closed form. Eliminate A. Gives linear 5-point FD stencil.

Model equation: $\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + 2\lambda\frac{\partial\omega}{\partial x} + 2\mu\frac{\partial\omega}{\partial y} = 0$

Allen - Southwell approximation:

$$\begin{bmatrix} & \alpha e^{\mu h} & \\ e^{-\lambda h} & 2\{\cosh(\lambda h) + \cosh(\mu h)\} & e^{\lambda h} \\ & \alpha e^{-\mu h} & \end{bmatrix} / h^2 = 0, \quad \alpha = \frac{\mu}{\lambda}\frac{\sinh(\lambda h)}{\sinh(\mu h)}$$

For h -> 0:

$$\begin{bmatrix} & 1 & \\ 1 & -2 & 1 \\ & 1 & \end{bmatrix} / h^2 + \begin{bmatrix} & \mu & \\ -\lambda & & \lambda \\ & -\mu & \end{bmatrix} / h = O(1)$$

For h large: Upwind directed 3-point formula

For all h : DIAGONALLY DOMINANT

+ Enhances convergence of relaxation-type methods
+ Prevents solutions from developing mesh-size oscillations
− Lower accuracy than centered scheme at finite h.

Many later refinements to upwinding:

⇨ Stronger diagonal dominance

⇨ Higher orders of formal accuracy

Chart 8. Allen and Southwell (1955)

1980 Fornberg

COMPUTATION

▷ **Replace relaxation methods by Newton's method; keep Finite Differences**

For Scalar Equations:

$f(x) = 0$

$$\begin{cases} x_0 \text{ Close Guess} \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \quad n = 0,1,2,\ldots \end{cases}$$

Can Write:

$$\underbrace{f'(x_n)}_{\text{Known}} \cdot \underbrace{\Delta x_n}_{\substack{\text{Update} \\ x_{n+1}-x_n}} = \underbrace{-f(x_n)}_{\substack{\text{Known,} \\ \text{Residual}}}$$

For Systems:

$f(x,y,z) = 0$
$g(x,y,z) = 0$
$h(x,y,z) = 0$

Similar Iteration:

$$\underbrace{\begin{bmatrix} \frac{\delta f}{\delta x} & \frac{\delta f}{\delta y} & \frac{\delta f}{\delta z} \\ \frac{\delta g}{\delta x} & \frac{\delta g}{\delta y} & \frac{\delta g}{\delta z} \\ \frac{\delta h}{\delta x} & \frac{\delta h}{\delta y} & \frac{\delta h}{\delta z} \end{bmatrix}}_{\substack{\text{Known,} \\ \text{"Jacobian"}}} \underbrace{\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}}_{\substack{\text{Update} \\ \text{Unknown}}} = - \underbrace{\begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix}}_{\substack{\text{Known,} \\ \text{Residual}}}$$

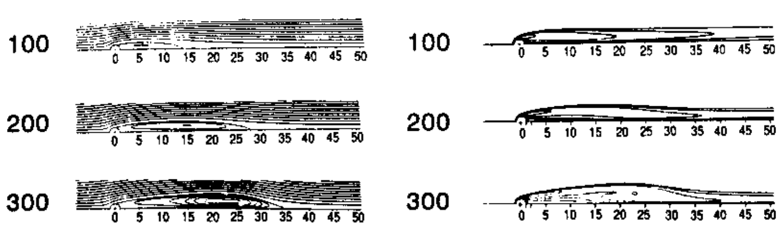
▷ **Quadratic convergence: On NxN-grid, costs $O(N^4)$ operations/iteration.**

▷ **No possibility to 'inherit' temporal instabilities into artificial time of numerical iterations**

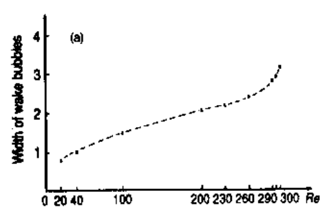
▷ **Easier to implement boundary conditions**

▷ **Centered, 2nd order FD instead of 'upwinding'**
Upwinding previously needed for two reasons:
- Enhance convergence
- Prevent mesh-size oscillations

STREAMFUNCTION — VORTICITY



WIDTH OF WAKE BUBBLE



LENGTH OF WAKE BUBBLE

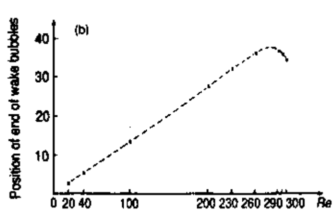
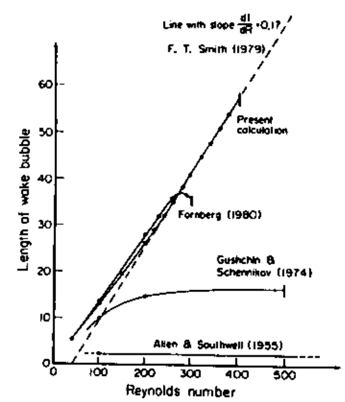


Chart 9. Fornberg (1980)

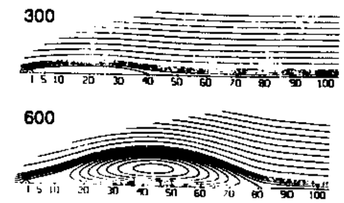
1983,1985, 1991 BF

COMPUTATIONS

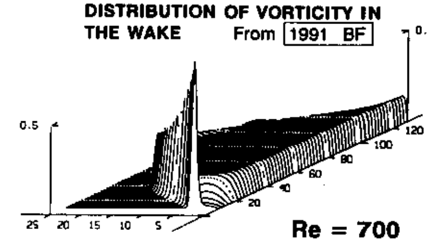
ERROR WITH DECREASING WAKE LENGTH CORRECTED
From 1983 BF



WAKES AT HIGH VALUES OF Re
From 1985 BF



DISTRIBUTION OF VORTICITY IN THE WAKE From 1991 BF



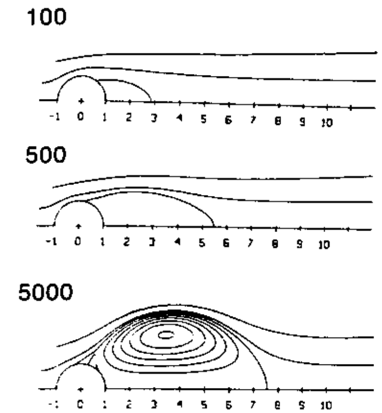
Re = 700

Chart 10. Fornberg (1983, 1985, 1991)

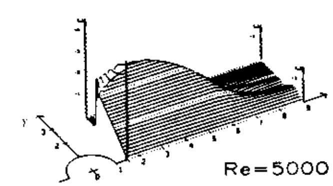
1988 BF FLOW PAST A SPHERE

COMPUTATION

STREAM FUNCTION



VORTICITY



DRAG COEFFICIENT

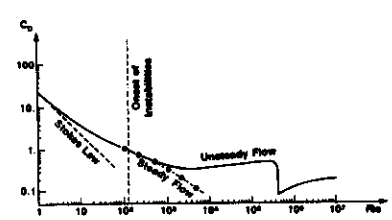
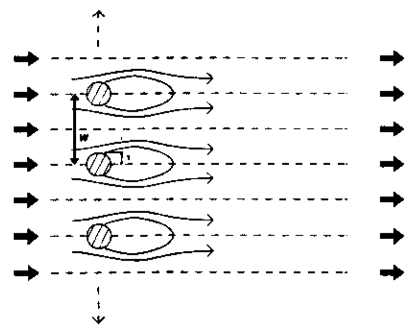


Chart 11. Fornberg (1988)

1991 BF

FLOW PAST A ROW OF CYLINDERS



COMPUTATION

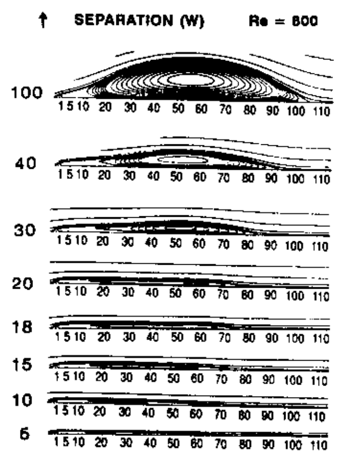


Chart 12. Fornberg (1991)

1992 Natarajan, BF, Acrivos

COMPUTATION

Newton's method - bi-quadratic Finite Elements

Re = 600 , W = 40

Distribution of vorticity in the wake

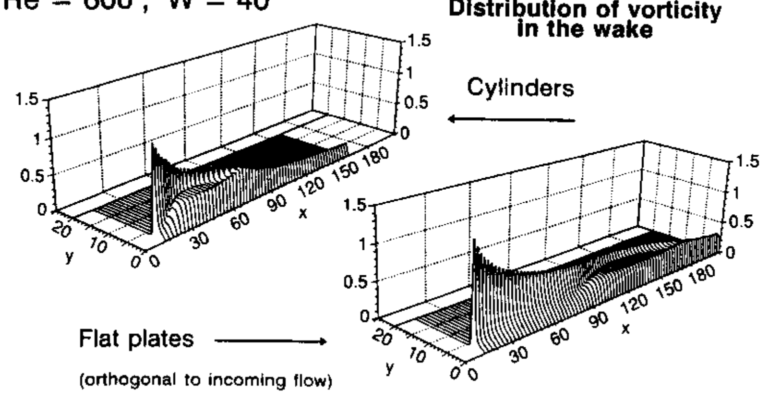


Chart 13. Natarajan, Fornberg, Acrivos (1992)

1991 Shroff, Keller

NUMERICAL METHOD

STABILIZATION OF UNSTABLE PROCEDURES: A HYBRID ALGORITHM FOR CONTINUATION

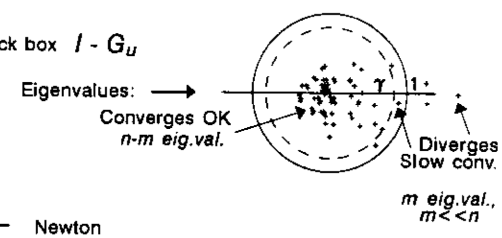
HAVE: Time dependent 'black box' code:

$$u^{(\nu+1)} = \boxed{u^{(\nu)} - G(u^{(\nu)}, R)}$$

(and nothing else; black box expensive - need to keep number of evaluations low)

WANT: To find steady-state solutions $G(u, R) = 0$ in regimes (for high R) where iterations diverge.

CONSIDER: Jacobian of black box $I - G_u$



IDEA: Decompose $\mathbb{R}^n$ into orthogonal subspaces:

$P\mathbb{R}^n$ dim m ← Newton
$Q\mathbb{R}^n$ dim $n-m$ ← Black box update

Use Krylov technique + QR to keep subspace current

POTENTIAL ADVANTAGES:

- Low cost per iteration
  size of linear system prop. to m (corr. to number of physically unstable modes) rather than n (corr. to number of grid points - typically many orders of magnitude larger)
- Sparsity pattern in Jacobian of no consequence
  method effective on unstructured grids and in 3-D
- Method well suited for parallel computing
  domain decomposition straightforward for computationally most expensive part (the 'black box' solver)

Chart 14. Shroff, Keller (1991)