Example of an ill-conditioned linear system - and of how the size of the residual need not give any information about the accuracy of an approximate solution

Let $\underset{\sim}{\tilde{x}} = \begin{bmatrix} -2.9067 \\ 9.4542 \\ -12.4152 \\ 26.9892 \\ -18.6606 \end{bmatrix}$ **be an approximate solution to** $\begin{bmatrix} 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \\ 1/6 & 1/7 & 1/8 & 1/9 & 1/10 \end{bmatrix} \begin{bmatrix} \\ \\ \underline{x} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0.882 \\ 0.744 \\ 0.618 \\ 0.521 \\ 0.447 \end{bmatrix}$.

Then the residual becomes (exactly) $A\underset{\sim}{\tilde{x}} - \underline{b} = \begin{bmatrix} -0.00001 \\ 0.00001 \\ -0.00001 \\ 0.00001 \\ -0.00001 \end{bmatrix}$.

Does this mean that $\underset{\sim}{\tilde{x}}$ **is a good approximation to the true solution?**

NO:

The true solution is $\underline{x} = \begin{bmatrix} -2.52 \\ 5.04 \\ 2.52 \\ 7.56 \\ -10.08 \end{bmatrix}$.

The difficulty comes from the fact that *A* has a very large condition number. In fact,

$$A^{-1} = \begin{bmatrix} 450 & -4200 & 12600 & -15120 & 6300 \\ -4200 & 44100 & -141120 & 176400 & -75600 \\ 12600 & -141120 & 470400 & -604800 & 264600 \\ -15120 & 176400 & -604800 & 793800 & -352800 \\ 6300 & -75600 & 264600 & 352800 & 158760 \end{bmatrix}$$

Eigenvalues of *A*:

1.055948840886970
0.082285410079855
0.003357591063700
0.000074136741258
0.000000687894883

$cond(A)_1 = 2.8172 \cdot 10^6$
$cond(A)_2 = 1.5350 \cdot 10^6$
$cond(A)_\infty = 2.8172 \cdot 10^6$