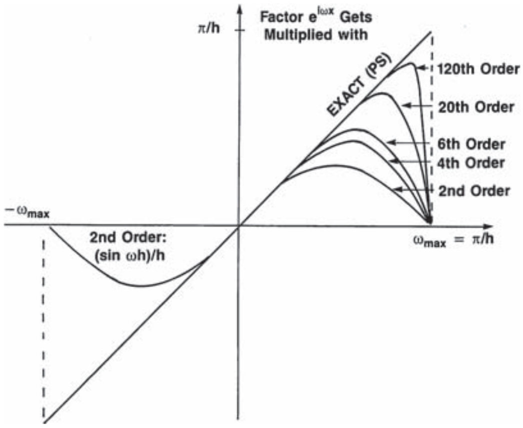

Figure 2.2. *Multiplicative factors arising when the pth order FD approximation for d/dx is applied to $e^{i\omega x}$. Reprinted with permission from Cambridge University Press. [90]*

the spatial discretization method approximates $\frac{d}{dx}e^{i\omega x}$. The exact result should be

$$\frac{d}{dx}e^{i\omega x} = i\omega e^{i\omega x}. \tag{2.2}$$

With centered second-order FD (abbreviated FD2), we get instead

$$D^{(2)}e^{i\omega x} = \frac{e^{i\omega(x+h)} - e^{i\omega(x-h)}}{2h} = i\,\frac{\sin \omega h}{h}\,e^{i\omega x} \tag{2.3}$$

and for FDp

$$D^{(p)}e^{i\omega x} = i\,\frac{\sin \omega h}{h}\left\{\sum_{k=0}^{p/2-1}\frac{(k!)^2}{(2k+1)!}\left(2\sin\frac{\omega h}{2}\right)^{2k}\right\}e^{i\omega x}. \tag{2.4}$$

Figure 2.2 displays the factors in front of $e^{i\omega x}$ in (2.2)–(2.4), omitting the "i."

While the PS method is constructed to be exact for every mode that can be represented on the grid, lower-order FD methods treat well only a rather narrow range surrounding $\omega = 0$. The order of accuracy p of a scheme matches the number of derivatives that are correct at $\omega = 0$. Analysis of these curves, together with knowledge about the spectral content of initial conditions, can therefore provide good estimates for how errors will grow during time integrations [85].

2.1.3.2 ▪ Convergence for nonsmooth functions

Gibbs phenomenon The best-known version of the Gibbs phenomenon is the overshoot that arises when a discontinuous function is represented by a truncated set of Fourier expansion terms. A similar situation occurs when a Fourier or a cubic spline interpolant is obtained by means of interpolation on an equispaced grid. Figure 2.3 illustrates these three cases. In the limits of increasingly many terms and of increasingly high node densities, respectively, the formulas for the peak heights are

a. Truncated Fourier series	$\frac{1}{2}+\frac{1}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi$	≈ 1.0895,
b. Fourier interpolation	$\max_{0<\xi<1}\left\{\frac{\sin \pi\xi}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\xi-k}\right\}$	≈ 1.1411,
c. Cubic spline interpolation	$\frac{1}{6}(8-2\sqrt{2}-\sqrt{3}-\sqrt{6})$	≈ 1.1078;