The following two theorems are quoted from the Hunter and Nachtergaele text:

**Theorem 5.19 (Bounded linear transformation)** Let $X$ be a normed linear space and $Y$ a Banach space. If $M$ is a dense linear subspace of $X$ and

$$T:M \subset X \rightarrow Y$$

is a bounded linear map, then there is a unique bounded linear map $\overline{T}:X \rightarrow Y$ such that $\overline{T}x = Tx$ for all $x \in M$. Moreover, $\|\overline{T}\| = \|T\|$.

**Theorem 5.58 (Hahn-Banach)** If $Y$ is a linear subspace of a normed linear space $X$ and $\psi:Y \rightarrow \mathbb{R}$ is a bounded linear functional on $Y$ with $\|\psi\| = M$, then there is a bounded linear functional $\varphi:X \rightarrow \mathbb{R}$ on $X$ such that $\varphi$ restricted to $Y$ is equal to $\psi$ and $\|\varphi\| = M$.

What are the key differences?

1. BLT allows the output to be any Banach space, while HB restricts to $\mathbb{R}$, i.e., BLT is for general linear transformations, HB is only for linear functionals (a specific type of linear transformation).

2. BLT requires a dense subspace, but guarantees a unique extension. HB doesn’t require a dense subspace, but also doesn’t guarantee a unique extension.

Our book doesn’t have the most general version of Hahn-Banach, so here is a more general version, but we don’t prove it (proof relies on Zorn’s lemma). Royden and Reed/Simon have proofs, for example.

**Theorem 1 (Hahn-Banach, general).** Let $X$ be a linear space over a field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). Let $p:X \rightarrow \mathbb{R}$ be a real-valued functional on $X$ satisfying

$$p(x + y) \leq p(x) + p(y), \quad \forall x, y \in X \quad \text{"sub-linear"}$$

$$p(\alpha x) = |\alpha|p(x), \quad \forall \alpha \in \mathbb{F}, \ x \in X \quad \text{"positive homogeneous"}.$$ 

Furthermore, let $Y \subset X$ be a subspace of $X$ and let $\psi:Y \rightarrow \mathbb{F}$ be a linear functional on $Z$ such that

$$|\psi(x)| \leq p(x), \quad \forall x \in Y.$$ 

Then $\psi$ has a linear extension $\varphi:X \rightarrow \mathbb{F}$ with

$$|\varphi(x)| \leq p(x), \quad \forall x \in X.$$ 

Note that sub-linearity implies $p(x) = 0$, and using this with the positive homogeneous property implies $p(x) \geq 0$ for all $x \in X$.

If $\psi$ is a bounded linear functional, then choosing $p(x) = \|\psi\| \cdot \|x\|$ shows that the general version implies the Thm. 5.58 version.