BLT vs Hahn Banach
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The following two theorems are quoted from the Hunter and Nachtergaele text:

Theorem 5.19 (Bounded linear transformation) Let X be a normed linear space and Y a Banach space. If M is a dense linear subspace of X and

$$T : M \subset X \to Y$$

is a bounded linear map, then there is a unique bounded linear map $\overline{T} : X \to Y$ such that $\overline{T}x = Tx$ for all $x \in M$. Moreover, $\|\overline{T}\| = \|T\|$.

Theorem 5.58 (Hahn-Banach) If Y is a linear subspace of a normed linear space X and $\psi : Y \to \mathbb{R}$ is a bounded linear functional on Y with $\|\psi\| = M$, then there is a bounded linear functional $\varphi : X \to \mathbb{R}$ on X such that φ restricted to Y is equal to ψ and $\|\varphi\| = M$.

What are the key differences?

1. BLT allows the output to be any Banach space, while HB restricts to $\mathbb{R}$, i.e., BLT is for general linear transformations, HB is only for linear functionals (a specific type of linear transformation).

2. BLT requires a dense subspace, but guarantees a unique extension. HB doesn't require a dense subspace, but also doesn't guarantee a unique extension.

Our book doesn't have the most general version of Hahn-Banach, so here is a more general version, but we don't prove it (proof relies on Zorn's lemma). Royden and Reed/Simon have proofs, for example.

Theorem 1 (**Hahn-Banach**, general)**.** *Let X be a linear space over a field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$). Let $p : X \to \mathbb{R}$ be a real-valued functional on X satisfying*

$$\begin{aligned} p(x+y) &\leq p(x) + p(y), \quad \forall\, x, y \in X \quad \textit{“sub-linear”} \\ p(\alpha x) &= |\alpha|\, p(x), \quad \forall\, \alpha \in \mathbb{F},\ x \in X \quad \textit{“positive homogeneous”}. \end{aligned}$$

Furthermore, let $Y \subset X$ be a subspace of X and let $\psi : Y \to \mathbb{F}$ be a linear functional on Z such that

$$|\psi(x)| \leq p(x), \quad \forall\, x \in Y.$$

Then ψ has a linear extension $\varphi : X \to \mathbb{F}$ with

$$|\varphi(x)| \leq p(x), \quad \forall\, x \in X.$$

Note that sub-linearity implies $p(x) = 0$, and using this with the positive homogeneous property implies $p(x) \geq 0$ for all $x \in X$.

If ψ is a *bounded* linear functional, then choosing $p(x) = \|\psi\| \cdot \|x\|$ shows that the general version implies the Thm. 5.58 version.