Preliminary Exam
Partial Differential Equations
1:30 - 4:30 PM, Fri. Jan. 10, 2019
Room: Newton Lab (ECCR 257)

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

Student ID:__________________________________

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Quasilinear first order equations.**
   Consider the Cauchy problem

$$\begin{aligned} u_t + (u+u^2)u_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= f(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

   (a) Suppose $f \in C^1(\mathbb{R})$ and f, f' are bounded functions. Prove that a continuously differentiable solution $u(x,t)$ to Eq. (1) exists and is unique for $x \in \mathbb{R}$, $t \in [0, t_*)$ for some $t_* > 0$.

   (b) Provide an additional, necessary condition on f for the solution to Eq. (1) to exist for all $t > 0$, i.e., for $u(x,t)$ to remain continuously differentiable for all $t > 0$.

   Solution:

   (a) Use the method of characteristics. Since

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \quad \text{along the characteristic curves } x(t) \text{ satisfying} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u + u^2,$$

   we can integrate each of these ODEs to obtain

$$u(x,t) = f(x_0), \quad \text{along} \quad x = (u+u^2)t + x_0.$$

   This is the implicit solution for u

$$u(x,t) = f(x - (u+u^2)t). \tag{2}$$

   By the boundedness of f and f', there exist $M, M' > 0$ such that

$$|f(x)| < M, \quad |f'(x)| < M', \quad x \in \mathbb{R}.$$

   In order to guarantee existence of a solution to this implicit equation, we define the mapping

$$F(u,x,t) = u - f(x - (u+u^2)t), \quad F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$$

and apply the implicit function theorem. By virtue of the continuous differentiability of f, $F \in C^1(\mathbb{R} \times \mathbb{R}^2)$. Applying the initial data, $F(f(x_0), x_0, 0) = 0$. We compute

$$\frac{\partial F}{\partial u} = 1 + f'(x_0)(1 + 2u)t.$$

Then, for $t < T = \frac{1}{(1+2M)M'}$, $F_u \neq 0$ and the implicit function theorem implies that there exists a unique $u \in C^1(\mathbb{R} \times [0, t_*))$ such that $F(u(x,t), x, t) = 0$ for some $0 < t_* < T$ as desired.

(b) The Cauchy problem (1) suffers from the possibility of finite time singularity formation in which $|u_x| \to \infty$ as $(x,t) \to (x_\mathrm{b}, t_\mathrm{b})$. To check when this occurs, we use the implicit form of the solution (2) to compute

$$\begin{aligned} u_x &= f'(x_0)(1 - t(1+2u)u_x) \\ \Rightarrow u_x &= \frac{f'(x_0)}{1 + t(1 + 2f(x_0))f'(x_0)}. \end{aligned}$$

Consequently, a necessary condition to ensure that u_x is finite for all $t > 0$ is the requirement that

$$(1 + 2f(x_0))f'(x_0) \geq 0, \ x_0 \in \mathbb{R} \quad \iff \quad \frac{\mathrm{d}}{\mathrm{d}x}(f(x) + f(x)^2) \geq 0, \ x \in \mathbb{R}.$$

In other words, $f(x) + f(x)^2$ must be a monotone increasing function of x.

2. **Heat Equation.**

Let $D = (0, L) \times (0, T]$ and assume that $u \in C(\bar{D}) \cap C^2(D)$ is a solution to

$$\begin{aligned} u_t(x,t) &= g(x)u_{xx}(x,t) + F(x,t), && 0 < x < L, \quad 0 < t \leq T. \\ u(x,0) &= f(x), && 0 < x < L, \\ u(0,t) &= r(t), && 0 < t \leq T, \\ u(L,t) &= s(t), && 0 < t \leq T, \end{aligned} \tag{3}$$

where $g(x) > 0$ for all $x \in (0, L)$.

(a) Let $B = \bar{D} \backslash D$. If $F \leq 0$, prove that

$$\max_{\bar{D}} u(x,t) = \max_{B} u(x,t).$$

(b) Prove that the solutions to Eq. (3) are unique.

Solution:

(a) First, let's prove the case $F < 0$. Suppose that the statement is not true. Then, the maximum of u would be attained at a point (x_0, t_0) with $0 < x_0 < L$,

$0 < t \leq T$. By basic calculus, we would have $u_t(x_0,t_0) \geq 0$, $u_x(x_0,t_0) = 0$, $u_{xx}(x_0,t_0) \leq 0$. Evaluating (3) at this point, we would have

$$F(x_0,t_0) = u_t(x_0,t_0) - g(x_0)u_{xx}(x_0,t_0) \geq 0,$$

which contradicts $F < 0$. When $F \leq 0$, we can define $u^\epsilon = u - \epsilon t$. Then $u^\epsilon_t = gu^\epsilon_{xx} - \epsilon$ and we can apply the previous case with $F - \epsilon < 0$ to get $\max_{\bar{D}} u^\epsilon(x,t) = \max_B u^\epsilon(x,t)$. Letting $\epsilon \to 0$, we obtain the desired result.

(b) Suppose there are two solutions to (3), u_1 and u_2. Then $v = u_1 - u_2$ would satisfy

$$\begin{aligned} v_t(x,t) &= g(x)v_{xx}(x,t), && 0 < x < L, \quad 0 < t \leq T. \\ v(x,0) &= 0, && 0 < x < L, \\ v(0,t) &= 0, && 0 < t \leq T, \\ v(L,t) &= 0, && 0 < t \leq T. \end{aligned}$$

For this system $v \equiv 0$ vanishes on B by applying part (a) to the IBVP for v and $-v$ implying $u_1 \equiv u_2$ proving uniqueness.

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3. **Wave Equation.** Consider the initial boundary value problem (IBVP):

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && x > 0,\ t > 0, \\ u(x,0) &= 0 && x > 0, \\ u_t(x,0) &= \psi(x) && x > 0, \\ u_x(0,t) &= 0 && t > 0. \end{aligned}$$

(a) Use an energy argument to prove the solutions to the above IBVP are unique, applying minimal assumptions on $u(x,t)$. State these minimal assumptions.

Solution: First define $w = u_1 - u_2$, which has corresponding IBVP

$$w_{tt} = c^2 w_{xx},\ x,t > 0; \quad w(x,0) \equiv w_t(x,0) \equiv w_x(0,t) \equiv 0,$$

and define the energy $E(t) = \frac{1}{2}\int_0^\infty w_t^2(x,t) + c^2 w_x^2(x,t)dx$, then

$$\begin{aligned} E'(t) &= \int_0^\infty w_t(x,t)w_{tt}(x,t) + c^2 w_{xt}(x,t)w_x(x,t)dx \\ &= \left[c^2 w_t(x,t)w_x(x,t)\right]_0^\infty + \int_0^\infty w_t(x,t)\left[w_{tt}(x,t) - c^2 w_{xx}(x,t)\right]dx \\ &= \int_0^\infty w_t(x,t) \cdot 0 dx = 0, \end{aligned}$$

assuming $u_t, u_x \to 0$ as $x \to \infty$ and using the Neumann BC. We also know $E(0) = \frac{1}{2}\int_0^\infty 0^2 + 0^2 dx = 0$, so $E(t) \equiv 0$. Assuming u_t and u_x are C^1 and L^2 integrable, this implies $w_t \equiv w_x \equiv 0$ implying $w \equiv 0$ (using $w(x,0) \equiv 0$), so $u_1 \equiv u_2$.

(b) Solve for $u(x,t)$. What assumptions on ψ are needed for a classical solution?

Solution: The PDE implies $(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$ so $u = F(x-ct)+G(x+ct)$, so when $x > ct$, u is not influenced by the boundary and d'Alembert's solution implies

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy,$$

but when $x < ct$, the boundary condition influences the solution. To see how, note

$$\begin{aligned} F(x) + G(x) &= 0 \\ -cF'(x) + cG'(x) = \psi(x) \quad &\Rightarrow \quad -F(x) + G(x) + A = \frac{1}{c}\int_0^x \psi(y)dy \\ F'(-ct) + G'(ct) = 0 \quad &\Rightarrow \quad -F(-x) + G(x) + B = 0. \end{aligned}$$

Subtracting the last two integrated equations at $x = 0$ implies $A = B$, whereas summing the first two equations implies

$$G(x) = \frac{1}{2}\left[\frac{1}{c}\int_0^x \psi(y)dy - A\right],$$

so that when $x < ct$, this along with the third equation imply

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) = G(ct-x) + A + G(x+ct) \\ &= \frac{1}{2c}\left[\int_0^{ct-x} \psi(y)dy + \int_0^{x+ct} \psi(y)dy\right]. \end{aligned}$$

We need that $\psi \in C^1$; and $\lim_{x\to 0^+} \psi(x) = 0$.

4. **Poisson's Equation/Green's Functions.**

Consider the problem

$$\begin{aligned} \Delta u(\mathbf{x}) &= f(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{4}$$

where $\Omega = \{\mathbf{x} = (x,y,z) \subseteq \mathbb{R}^3 : z > 0, \|\mathbf{x}\|_2 < R\}$.

(a) Construct an appropriate Green's function for this problem.

Solution: Leveraging the fundamental solution $\Phi(\mathbf{x}-\mathbf{y}) = 1/(4\pi|\mathbf{x}-\mathbf{y}|)$ to $-\Delta u = \delta(\mathbf{x}-\mathbf{y})$ and defining $\hat{\mathbf{x}} = (x,y,-z)$, $\bar{\mathbf{x}} = R^2\mathbf{x}/|\mathbf{x}|^2$, and $\tilde{\mathbf{x}} = R^2\hat{x}/|\mathbf{x}|^2$, we write

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi(\hat{\mathbf{x}}-\mathbf{y}) - \Phi(\frac{|\mathbf{x}|}{R}|\bar{\mathbf{x}}-\mathbf{y}|) + \Phi(\frac{|\mathbf{x}|}{R}|\tilde{\mathbf{x}}-\mathbf{y}|),$$

which vanishes along $\partial\Omega$ as along the sphere surface $\mathbf{x} = \bar{\mathbf{x}}$ and $\hat{\mathbf{x}} = \tilde{\mathbf{x}}$ and along $z = 0$ we have $\mathbf{x} = \hat{\mathbf{x}}$ and $\bar{\mathbf{x}} = \tilde{\mathbf{x}}$.

(b) Using the Green's function found in (a), construct an explicit formula for the solution in terms of the functions f and g.

Solution: Using Green's theorem, we find

$$\int_\Omega (u(\mathbf{y})\Delta G(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y})\Delta u(\mathbf{y}))\, d\mathbf{y} = \int_{\partial\Omega} \left(u(\mathbf{y})\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial n}(\mathbf{y})\right) dS_{\mathbf{y}}$$

$$-u(\mathbf{x}) - \int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y} = \int_{\partial\Omega} \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y})g(\mathbf{y})d\mathbf{y}$$

$$u(\mathbf{x}) = -\int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y} - \int_{\partial\Omega} \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y})g(\mathbf{y})d\mathbf{y}.$$

5. **Separation of Variables.**
Consider the following IBVP for the heat equation

$$\begin{aligned} u_t &= ku_{xx}, && x \in (0,1), \quad t > 0, \quad k > 0, \\ u(x,0) &= f(x), && x \in (0,1), \\ u_x(0,t) &= u(0,t), && t > 0, \\ u_x(1,t) &= -u(1,t), && t > 0. \end{aligned}$$

(a) Assuming separated solutions $u(x,t) = X(x)T(t)$, derive the boundary value problem for $X(x)$, and show it is a symmetric Sturm-Liouville problem.

Solution: Plugging the ansatz into the PDE, we have $XT' = kX''T$ implying

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \quad \Rightarrow \quad T' = -\lambda kT; \quad X'' = -\lambda X,$$

with boundary conditions $X'(0) = X(0)$ and $X'(1) = -X(1)$, so by assuming $X, Y \in L^2(0,1)$ with $Y'(0) = Y(0)$ and $Y'(1) = -Y(1)$:

$$\begin{aligned} \langle X'', Y\rangle &= \int_0^1 X''(x)\bar{Y}(x)dx = X'(1)\bar{Y}(1) - X'(0)\bar{Y}(0) - \int_0^1 X'(x)\bar{Y}'(x)dx \\ &= X'(1)\bar{Y}(1) - X'(0)\bar{Y}(0) - X(1)\bar{Y}'(1) + X(0)\bar{Y}'(0) + \int_0^1 X(x)\bar{Y}''(x)dx \\ &= -X(1)\bar{Y}(1) - X(0)\bar{Y}(0) + X(1)\bar{Y}(1) + X(0)\bar{Y}(0) + \int_0^1 X(x)\bar{Y}''(x)dx \\ &= \int_0^1 X(x)\bar{Y}''(x)dx = \langle X, Y''\rangle \end{aligned}$$

(b) Solve for the general form of the solution $u(x,t)$. Make sure you show all eigensolutions must decay in time, and do not blow up as $t \to 0$.

Solution: Starting with the BVP for $X(x)$, we note when $\lambda = -\nu^2 < 0$, solutions are of the form $X(x) = A_\nu e^{\nu x} + B_\nu e^{-\nu x}$, so Robin BCs imply

$$X(0) = A_\nu + B_\nu = \nu(A_\nu - B_\nu) = X'(0) \quad \to \quad B_\nu = \frac{\nu-1}{\nu+1}A_\nu$$

and

$$\begin{aligned} &X(1) = A_\nu\left[e^\nu + \frac{\nu-1}{\nu+1}e^{-\nu}\right] = -A_\nu\nu\left[e^\nu - \frac{\nu-1}{\nu+1}e^{-\nu}\right] = -X'(1) \\ &\to \quad (\nu+1)e^\nu = \frac{(\nu-1)^2}{\nu+1}e^{-\nu} \quad \to \quad e^{2\nu} = \frac{(\nu-1)^2}{(\nu+1)^2} \quad \to \quad \nu = 0, \end{aligned}$$

and for $\lambda = 0$ and $X(x) = A + Bx$, we have $X(0) = A = B = X'(0)$ and $X(1) = 2A = -A = -X'(1)$ implies $A = 0$, so the only nontrivial solutions are of form $X(x) = A_\nu \sin(\nu x) + B_\nu \cos(\nu x)$ with $\lambda = \nu^2 > 0$, so the Robin BCs imply

$$X(0) = B_\nu = \nu A_\nu = X'(0)$$

and

$$X(1) = A_\nu(\sin\nu + \nu\cos\nu) = -\nu A_\nu(\cos\nu - \nu\sin\nu) = -X'(1) \quad \to \quad \tan\nu = \frac{2\nu}{\nu^2 - 1}$$

which has an infinite set of solutions $\nu_{1,2,3,\ldots}$, since the RHS is decreasing and positive for $\nu > 3$ while $\tan\nu$ is periodic and ranges from $(-\infty, \infty)$. Moreover, this is a symmetric SL problem, so it is guaranteed to have a complete orthonormal set of eigenfunctions.
Clearly $\nu = 0$ is a solution, but this yields the trivial $X(x) \equiv 0$.
Defining $X_j(x) = A_j(\sin(\nu_j x) + \nu_j \cos(j\pi x)) = A_j\phi_j(x)$ and $\lambda_j = \nu_j^2$, so $T_j(t) = \mathrm{e}^{-\nu_j^2 kt}$, and

$$u(x,t) = \sum_{j=1}^{\infty} A_j \mathrm{e}^{-\nu_j^2 kt}\phi_j(x), \qquad A_j = \frac{\langle \phi_j(x), f(x)\rangle}{\langle \phi_j(x), \phi_j(x)\rangle}.$$

(c) State minimal assumptions on $f(x)$ needed so $u(x,t) \in C^2$ on $x \in (0,1)$ and $t > 0$.
Solution: We need only that $f(x)$ be L^2-integrable, so A_j can be defined and the series converges for all $x \in (0,1)$ due to the exponential term.