Preliminary Exam
Partial Differential Equations
1:30 - 4:30 PM, Fri. Jan. 10, 2019
Room: Newton Lab (ECCR 257)

Student ID:____________________________________

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Quasilinear first order equations.**
   Consider the Cauchy problem

$$\begin{aligned} u_t + (u + u^2)u_x &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= f(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

   (a) Suppose $f \in C^1(\mathbb{R})$ and f, f' are bounded functions. Prove that a continuously differentiable solution $u(x,t)$ to Eq. (1) exists and is unique for $x \in \mathbb{R}$, $t \in [0, t_*)$ for some $t_* > 0$.

   (b) Provide an additional, necessary condition on f for the solution to Eq. (1) to exist for all $t > 0$, i.e., for $u(x,t)$ to remain continuously differentiable for all $t > 0$.

2. **Heat Equation.**
   Let $D = (0, L) \times (0, T]$ and assume that $u \in C(\bar{D}) \cap C^2(D)$ is a solution to

$$\begin{aligned} u_t(x,t) &= g(x)u_{xx}(x,t) + F(x,t), \qquad 0 < x < L, \quad 0 < t \le T. \\ u(x,0) &= f(x), \qquad 0 < x < L, \\ u(0,t) &= r(t), \qquad 0 < t \le T, \\ u(L,t) &= s(t), \qquad 0 < t \le T, \end{aligned} \tag{2}$$

   where $g(x) > 0$ for all $x \in (0, L)$.

   (a) Let $B = \bar{D} \backslash D$. If $F \le 0$, prove that

$$\max_{\bar{D}} u(x,t) = \max_{B} u(x,t).$$

   (b) Prove that the solutions to Eq. (3) are unique.

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3. **Wave Equation.** Consider the initial boundary value problem (IBVP):

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} && x > 0,\ t > 0,\\
u(x,0) &= 0 && x > 0,\\
u_t(x,0) &= \psi(x) && x > 0,\\
u_x(0,t) &= 0 && t > 0.
\end{aligned}
$$

(a) Use an energy argument to prove the solutions to the above IBVP are unique, applying minimal assumptions on $u(x,t)$. State these minimal assumptions.

(b) Solve for $u(x,t)$. What assumptions on ψ are needed for a classical solution?

4. **Poisson's Equation/Green's Functions.**
Consider the problem

$$
\begin{aligned}
\Delta u(\mathbf{x}) &= f(\mathbf{x}), && \mathbf{x} \in \Omega,\\
u(\mathbf{x}) &= g(\mathbf{x}), && \mathbf{x} \in \partial\Omega,
\end{aligned}
\tag{3}
$$

where $\Omega = \{\mathbf{x} = (x,y,z) \subseteq \mathbb{R}^3 : z > 0, \|\mathbf{x}\|_2 < R\}$.

(a) Construct an appropriate Green's function for this problem.

(b) Using the Green's function found in (a), construct an explicit formula for the solution in terms of the functions f and g.

5. **Separation of Variables.**
Consider the following IBVP for the heat equation

$$
\begin{aligned}
u_t &= k u_{xx}, && x \in (0,1),\quad t > 0,\quad k > 0,\\
u(x,0) &= f(x), && x \in (0,1),\\
u_x(0,t) &= u(0,t), && t > 0,\\
u_x(1,t) &= -u(1,t), && t > 0.
\end{aligned}
$$

(a) Assuming separated solutions $u(x,t) = X(x)T(t)$, derive the boundary value problem for $X(x)$, and show it is a symmetric Sturm-Liouville problem.

(b) Solve for the general form of the solution $u(x,t)$. Make sure you show all eigensolutions must decay in time, and do not blow up as $t \to 0$.

(c) State minimal assumptions on $f(x)$ needed so $u(x,t) \in C^2$ on $x \in (0,1)$ and $t > 0$.