1. Quasilinear first order equations.

Consider the Cauchy problem

\[ \begin{align*}
  u_t + (u + u^2)u_x &= 0, & x \in \mathbb{R}, & t > 0, \\
  u(x, 0) &= f(x), & x \in \mathbb{R}.
\end{align*} \tag{1} \]

(a) Suppose \( f \in C^1(\mathbb{R}) \) and \( f, f' \) are bounded functions. Prove that a continuously differentiable solution \( u(x, t) \) to Eq. (1) exists and is unique for \( x \in \mathbb{R}, t \in [0, t_*) \) for some \( t_* > 0 \).

(b) Provide an additional, necessary condition on \( f \) for the solution to Eq. (1) to exist for all \( t > 0 \), i.e., for \( u(x, t) \) to remain continuously differentiable for all \( t > 0 \).


Let \( D = (0, L) \times (0, T] \) and assume that \( u \in C(\bar{D}) \cap C^2(D) \) is a solution to

\[ \begin{align*}
  u_t(x, t) &= g(x)u_{xx}(x, t) + F(x, t), & 0 < x < L, & 0 < t \leq T, \\
  u(x, 0) &= f(x), & 0 < x < L, \\
  u(0, t) &= r(t), & 0 < t \leq T, \\
  u(L, t) &= s(t), & 0 < t \leq T,
\end{align*} \tag{2} \]

where \( g(x) > 0 \) for all \( x \in (0, L) \).

(a) Let \( B = \bar{D}\setminus D \). If \( F \leq 0 \), prove that

\[ \max_D u(x, t) = \max_B u(x, t). \]

(b) Prove that the solutions to Eq. (3) are unique.
3. **Wave Equation.** Consider the initial boundary value problem (IBVP):

\[
    \begin{align*}
        u_{tt} &= c^2 u_{xx} & x > 0, & t > 0, \\
        u(x, 0) &= 0 & x > 0, \\
        u_t(x, 0) &= \psi(x) & x > 0, \\
        u_x(0, t) &= 0 & t > 0.
    \end{align*}
\]

(a) Use an energy argument to prove the solutions to the above IBVP are unique, applying minimal assumptions on \(u(x, t)\). State these minimal assumptions.

(b) Solve for \(u(x, t)\). What assumptions on \(\psi\) are needed for a classical solution?

4. **Poisson’s Equation/Green’s Functions.**

Consider the problem

\[
    \begin{align*}
        \Delta u(x) &= f(x), & x \in \Omega, \\
        u(x) &= g(x), & x \in \partial \Omega,
    \end{align*}
\]

where \(\Omega = \{x = (x, y, z) \subseteq \mathbb{R}^3 : z > 0, \|x\|_2 < R\}\).

(a) Construct an appropriate Green’s function for this problem.

(b) Using the Green’s function found in (a), construct an explicit formula for the solution in terms of the functions \(f\) and \(g\).

5. **Separation of Variables.**

Consider the following IBVP for the heat equation

\[
    \begin{align*}
        u_t &= ku_{xx}, & x \in (0, 1), & t > 0, & k > 0, \\
        u(x, 0) &= f(x), & x \in (0, 1), \\
        u_x(0, t) &= u(0, t), & t > 0, \\
        u_x(1, t) &= -u(1, t), & t > 0.
    \end{align*}
\]

(a) Assuming separated solutions \(u(x, t) = X(x)T(t)\), derive the boundary value problem for \(X(x)\), and show it is a symmetric Sturm-Liouville problem.

(b) Solve for the general form of the solution \(u(x, t)\). Make sure you show all eigen-solutions must decay in time, and do not blow up as \(t \to 0\).

(c) State minimal assumptions on \(f(x)\) needed so \(u(x, t) \in C^2\) on \(x \in (0, 1)\) and \(t > 0\).