Preliminary Exam
Partial Differential Equations
9 AM - 12 PM, Fri. Jan. 11, 2019
Room: Newton Lab (ECCR 257)

Student ID:____________________________________

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points. A sheet of convenient formulae is provided.

1. **Method of characteristics.** Solve the following initial-boundary value problem:

$$\begin{aligned} u_t + \mathrm{e}^{-x} u_x &= \mathrm{e}^x, && x > 0, \quad t > 0, \\ u(x,0) &= f(x), && x > 0, \\ u(0,t) &= g(t), && t > 0. \end{aligned}$$

You will need to separate the domain into two regions, and be sure to identify the boundary between the two regions.

Solution: First consider the solution emanating from $x_0(s) = s$, $t_0(s) = 0$, $z_0(s) = f(s)$ (along the $t = 0$ boundary):

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= \mathrm{e}^{-x}, \quad x_0(s) = s, \\ \frac{\mathrm{d}t}{\mathrm{d}\tau} &= 1, \quad t_0(s) = 0, \\ \frac{\mathrm{d}z}{\mathrm{d}\tau} &= \mathrm{e}^{x}, \quad z_0(s) = f(s). \end{aligned}$$

We guarantee a unique solution by checking the Jacobian is nonzero along the initial curve:

$$J = \begin{vmatrix} 1 & \mathrm{e}^{-s} \\ 0 & 1 \end{vmatrix} = 1 > 0.$$

Integrating the t equation, we find $t = \tau$. Next, integrating the x equation, we find

$$\mathrm{e}^x = \tau + \mathrm{e}^s$$

This we can plug into the z equation

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \tau + \mathrm{e}^s \quad \Rightarrow \quad z = \frac{\tau^2}{2} + \mathrm{e}^s \tau + f(s).$$

Inverting our expressions for x and t, we find

$$u(x,t) = t\mathrm{e}^x - \frac{t^2}{2} + f\left[\ln|e^x - t|\right],$$

where we require $\mathrm{e}^x + 1 > t$ since the argument of $f(s)$ must be positive.

The solution emanating from $x = 0$ is parameterized by $x_0(s) = 0$, $t_0(s) = s$, $z_0(s) = g(s)$:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \mathrm{e}^{-x}, \quad x_0(s) = 0, \\
\frac{\mathrm{d}t}{\mathrm{d}\tau} &= 1, \quad t_0(s) = s, \\
\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \mathrm{e}^{x}, \quad z_0(s) = g(s).
\end{aligned}$$

We guarantee a unique solution by checking the Jacobian is nonzero along the initial curve:

$$J = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 < 0.$$

Integrating the t equation, we find $t = \tau + s$. Next, integrating the x equation, we find

$$\mathrm{e}^x = \tau + 1.$$

Plugging into the z equation, we find

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \tau + 1 \quad \Rightarrow \quad z = \frac{\tau^2}{2} + 2\tau + g(s).$$

Inverting our expressions for x and t, we find

$$u(x,t) = \frac{(\mathrm{e}^x - 1)^2}{2} + \mathrm{e}^x - 1 + g\left[t + 1 - \mathrm{e}^x\right]$$

where we require $t + 1 > \mathrm{e}^x$ since the argument of $g(s)$ must be positive. Ultimately:

$$u(x,t) = \begin{cases} t\mathrm{e}^x - \frac{t^2}{2} + f\left[\ln|e^x - t|\right], & \mathrm{e}^x > t + 1 \\ \frac{(\mathrm{e}^x-1)^2}{2} + \mathrm{e}^x - 1 + g\left[t + 1 - \mathrm{e}^x\right], & t + 1 > \mathrm{e}^x. \end{cases}$$

2. **Heat Equation.**

(a) Use energy methods to show the following initial boundary value problem has at most one solution:

$$\begin{aligned}
u_t(\mathbf{x},t) &= f(t)\Delta u(\mathbf{x},t), && f(t) > 0,\ \mathbf{x} \in \Omega, \\
u(\mathbf{x},0) &= g(\mathbf{x}), && \mathbf{x} \in \Omega \\
u(\mathbf{x},t) &= h(\mathbf{x},t) && \mathbf{x} \in \partial\Omega, \quad t > 0.
\end{aligned}$$

Assume $f(t) \in C^\infty[0,\infty)$, $g(\mathbf{x}) \in C^\infty(\partial\Omega)$, and $h(\mathbf{x},t) \in C^\infty(\partial\Omega \times [0,\infty))$ are all integrable functions and the domain Ω is simply connected.

Solution: Assume there are two solutions u_1 and u_2, and $w = u_1 - u_2$, then $w_t(\mathbf{x},t) = f(t)\nabla^2 w(\mathbf{x},t)$ and $w(\mathbf{x},0) = 0$ ($\mathbf{x} \in \Omega$) and $w(\mathbf{x},t) = 0$ on $\mathbf{x} \in \partial\Omega$. Now set

$$E(t) \equiv \int_\Omega w^2(\mathbf{x},t)\mathrm{d}\mathbf{x} \geq 0,$$

since the integrand is nonnegative, so then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^2(\mathbf{x},t)\mathrm{d}x = \int_\Omega 2w(\mathbf{x},t)w_t(\mathbf{x},t)\mathrm{d}\mathbf{x} \\ &= 2f(t)\int_\Omega w(\mathbf{x},t)\Delta w(\mathbf{x},t)\mathrm{d}\mathbf{x} = -2f(t)\int_\Omega |\nabla w(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \leq 0,\end{aligned}$$

where we have applied the divergence theorem, homogeneous boundaries, and the fact that $f(t) > 0$. Also, since $w(\mathbf{x},0) \equiv 0$, then $E(0) = 0$, and we can conclude $E \equiv 0$, implying $w \equiv 0$ so $u_1 \equiv u_2$ and there can be no more than one solution.

(b) Find the solution to the heat equation on the n-dimensional half space:

$$\begin{aligned} u_t(\mathbf{x},t) &= \kappa\Delta u(\mathbf{x},t), && \mathbf{x} \in \Omega \equiv \{\mathbf{x} \in \mathbb{R}^n | x_n > 0\}, \quad t > 0, \\ u(\mathbf{x},0) &= f(\mathbf{x}), && \mathbf{x} \in \Omega, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x},t) &= 0, && \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x},t) &= 0, && t > 0, \end{aligned}$$

where $\kappa > 0$ is a positive scalar, $f(\mathbf{x})$ is continuous and L^2 integrable on $\mathbb{R}^n$, and $\mathbf{n}$ is the unit normal to the boundary $\partial\Omega$.

Solution: The fundamental solution is

$$\Phi(\mathbf{x},t) = \frac{1}{(4\pi\kappa t)^{n/2}}\mathrm{e}^{-|\mathbf{x}|^2/(4\kappa t)},$$

so, defining $\tilde{\mathbf{x}} = (x_1, x_2, ..., -x_n)$, we can use even reflection to write

$$u(\mathbf{x},t) = \frac{1}{(4\pi\kappa t)^{n/2}}\int_\Omega \left[\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/(4\kappa t)} + \mathrm{e}^{-|\mathbf{x}-\tilde{\mathbf{y}}|^2/(4\kappa t)}\right] f(\mathbf{y})\mathrm{d}\mathbf{y}.$$

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3. **Wave Equation.** Consider the equation

$$\begin{aligned} u_{tt}(\mathbf{x},t) &= c^2\Delta u(\mathbf{x},t), \\ u(\mathbf{x},0) &= f(\mathbf{x}), && \mathbf{x} \in \mathbb{R}^3 \\ u_t(\mathbf{x},0) &= g(\mathbf{x}), && \mathbf{x} \in \mathbb{R}^3 \end{aligned}$$

where $c > 0$ is a positive scalar, and $f(\mathbf{x})$ and $g(\mathbf{x})$ are rapidly decaying, C^∞, and L^2 integrable functions.

(a) Find the equation the average of u:

$$\overline{u}(r,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(\mathbf{x},t)\sin\phi d\phi\ d\theta$$

satisfies where $\mathbf{x} = (x, y, z)$ and $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi\ \theta$, such that θ and ϕ are angles in spherical coordinates.

Solution: Averaging $u_{tt}(\mathbf{x}, t)$ as such, we simply obtain $\bar{u}_{tt}(r, t)$. Thus, to obtain the new PDE, we integrate the right hand side of the PDE, which we can write in spherical coordinates

$$\begin{aligned} c^2\overline{\Delta u(\mathbf{x},t)} &= \frac{c^2}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right]\sin\phi \mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{c^2}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\bar{u}}{\partial r}), \end{aligned}$$

where the second term vanishes because $\sin\pi = \sin 0 = 0$ and the third term vanishes due to periodicity in the spherical coordinate ρ, so

$$\bar{u}_{tt}(r,t) = \frac{c^2}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\bar{u}}{\partial r}(r,t)) = c^2\left[\frac{\partial^2\bar{u}}{\partial r^2}(r,t) + \frac{2}{r}\frac{\partial\bar{u}}{\partial r}(r,t)\right].$$

(b) Assume spherical symmetry of initial conditions ($f = f(r), g = g(r)$) and the solution $u = u(r,t)$, where $r \equiv |\mathbf{x}|$, and write the initial/boundary value problem for the radially symmetric function $v(r,t) = ru(r,t)$.

Solution: In this case,

$$v_{tt} = ru_{tt} = c^2\left[r\frac{\partial^2 u}{\partial r^2}(r,t) + 2\frac{\partial u}{\partial r}(r,t)\right] = c^2\frac{\partial}{\partial r}\left[\left(r\frac{\partial u}{\partial r}\right) + u\right] = c^2\frac{\partial^2}{\partial r^2}[ru] = c^2 v_{rr}.$$

Also, $v(r,0) = ru(r,0) = rf(r)$ and $v_t(r,0) = rg(r)$, similarly, so

$$\begin{aligned} v_{tt}(r,t) &= c^2 v_{rr}(r,t), \quad r\in(0,\infty),\ t\in(0,\infty) \\ v(0,t) &= 0, \qquad t\in(0,\infty) \\ v(r,0) &= rf(r), \quad r\in(0,\infty) \\ v_t(r,0) &= rg(r), \quad r\in(0,\infty) \end{aligned}$$

(c) Find the solution $v(r,t)$ and hence the solution $u(r,t)$.

Solution: Using d'Alembert's formula for the half line with homogeneous Dirichlet boundaries, using odd extension, we define $F_{\text{odd}}(r) = rf(r)$ for $r > 0$ and $F_{\text{odd}}(-r) = -F_{\text{odd}}(r)$, similarly $G_{\text{odd}}(r) = rg(r)$ for $r > 0$ and $G_{\text{odd}}(-r) = -G_{\text{odd}}(r)$, so

$$v(r,t) = \frac{1}{2}(F_{\text{odd}}(r+ct) + F_{\text{odd}}(r-ct)) + \frac{1}{2c}\int_{r-ct}^{r+ct} G_{\text{odd}}(s)\mathrm{d}s,$$

which can be written piecewise as

$$v(r,t) = \begin{cases} \frac{1}{2}((r+ct)f(r+ct) - (ct-r)f(ct-r)) + \frac{1}{2c}\int_{ct-r}^{ct+r} sg(s)\mathrm{d}s, & 0 < r < ct \\ \frac{1}{2}((r+ct)f(r+ct) + (r-ct)f(r-ct)) + \frac{1}{2c}\int_{r-ct}^{r+ct} sg(s)\mathrm{d}s, & r \geq ct. \end{cases}$$

Lastly, we change variables back to $u(r,t)$ to obtain

$$u(r,t) = \begin{cases} \frac{1}{2r}((r+ct)f(r+ct) - (ct-r)f(ct-r)) + \frac{1}{2cr}\int_{ct-r}^{ct+r} sg(s)\mathrm{d}s, & 0 < r < ct \\ \frac{1}{2r}((r+ct)f(r+ct) + (r-ct)f(r-ct)) + \frac{1}{2cr}\int_{r-ct}^{r+ct} sg(s)\mathrm{d}s, & r \geq ct. \end{cases}$$

4. **Green's Functions.**

(a) Consider Poisson's equation on the tilted half plane

$$\begin{aligned} \Delta u &= f(\mathbf{x}), & \mathbf{x} \in \Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 > 0\}, \\ u(\mathbf{x}) &= g(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{aligned} \tag{1}$$

Write the associated Green's function $G_H(\mathbf{x},\mathbf{y})$ using the method of images, and verify its corresponding boundary value problem.

Solution: Using the method of images, note $\mathbf{x}_H = (-x_2, -x_1)$ maps Ω to $\bar{\Omega}_c$, since for $x_1 + x_2 > 0$, then $-x_2 - x_1 < 0$ and vice versa, so

$$G_H(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi(\mathbf{x}_H - \mathbf{y}).$$

The corresponding boundary value problem is:

$$\begin{aligned} \Delta_{\mathbf{x}} G_H(\mathbf{x},\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}), & \mathbf{x},\mathbf{y} \in \Omega, \\ G_H(\mathbf{x},\mathbf{y}) &= 0, & \mathbf{x} \in \partial\Omega,\ \mathbf{y} \in \Omega. \end{aligned}$$

since $\Delta_{\mathbf{x}}\Phi(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$ since it is the fundamental solution to Laplace's equation, and $\Delta_{\mathbf{x}}\phi^y(\mathbf{x}) = 0$ on $\mathbf{y} \in \Omega$ since $\mathbf{x} \in \Omega$ implies $\mathbf{x}_H \notin \Omega$, then $\Delta_{\mathbf{x}} G_H(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$. Also, when $\mathbf{x} \in \partial\Omega$, then $x_1 = -x_2$ and $x_2 = -x_1$ and $\mathbf{x} = \mathbf{x}_H$ and $G_H(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi(\mathbf{x}-\mathbf{y}) = 0$.

(b) Consider Poisson's equation on the tilted half disc:

$$\begin{aligned} \Delta u &= f(\mathbf{x}), & \mathbf{x} \in \Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 > 0 \ \&\ |\mathbf{x}| < 1\}, \\ u(\mathbf{x}) &= 0 & \mathbf{x} \in \partial\Omega. \end{aligned} \tag{2}$$

Determine the associated Green's function $G_S(\mathbf{x},\mathbf{y})$ and show it satisfies the needed boundary conditions. Then, write the solution to Eq. (2) in terms of this Green's function, and show it satisfies $u(\mathbf{x}) = 0$ on $\mathbf{x} \in \partial\Omega$.

Solution: Define $\tilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|^2$ and $\tilde{\mathbf{x}}_H = \mathbf{x}_H/|\mathbf{x}|^2$, then

$$G_S(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi(\mathbf{x}_H - \mathbf{y}) - \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) + \Phi(|\mathbf{x}|(\tilde{\mathbf{x}}_H - \mathbf{y})).$$

When $\mathbf{x} = (x_1, -x_1) \in \partial\Omega$, then we have

$$G_S(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Phi(\mathbf{x} - \mathbf{y}) + \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) - \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) = 0,$$

and when $|\mathbf{x}| = 1$ with $\mathbf{x} \in \partial\Omega$, then $\tilde{\mathbf{x}} = \mathbf{x}$ and $\tilde{\mathbf{x}}_H = \mathbf{x}_H$ and $|\mathbf{x}_H| = 1$, so

$$G_S(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Phi(\mathbf{x} - \mathbf{y}) + \Phi(\mathbf{x}_H - \mathbf{y}) - \Phi(\mathbf{x}_H - \mathbf{y}) = 0,$$

so $G_Q(\mathbf{x}, \mathbf{y}) \equiv 0$ on $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} \in \Omega$.
Lastly, note that the solution to Eq. (2) in terms of G_S is

$$u(\mathbf{x}) = \int_\Omega G_S(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) \mathrm{d}\mathbf{y},$$

so when $\mathbf{x} \in \partial\Omega$ then

$$u(\mathbf{x}) = \int_\Omega 0 \cdot f(\mathbf{y}) \mathrm{d}\mathbf{y} = 0.$$

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5. **Separation of Variables.** Consider the forced, damped wave equation

$$u_{tt} = u_{xx} - 2\gamma u_t + e^{-x}, \quad 0 \le x \le L, \quad t > 0,$$

with $\gamma > 0$ and boundary and initial conditions

$$\begin{aligned} &u(0,t) = u(L,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad x \in (0,L), \\ &u_t(x,0) = 0, \quad x \in (0,L). \end{aligned}$$

Using separation of variables, find a *formal* solution $u(x,t)$ to the boundary value problem in terms of the function $f(x)$. (You can assume $\frac{\gamma L}{\pi}$ is not an integer.)
Solution: First, we remove the inhomogeneous term e^{-x} by defining $v(x,t) = u(x,t) + e^{-x} - 1 - \frac{x}{L}(e^{-L} - 1)$. We have

$$\begin{aligned} &v_t = u_t, \\ &v_{tt} = u_{tt}, \\ &v_{xx} = u_{xx} + e^{-x}, \\ &v(0,t) = u(0,t) = 0, \\ &v(L,t) = u(L,t) = 0. \end{aligned}$$

So we obtain the boundary value problem

$$\begin{aligned} &v_{tt} = v_{xx} - 2\gamma v_t, \quad 0 \le x \le L, \quad t > 0, \\ &v(0,t) = v(L,t) = 0, \quad t > 0, \\ &v(x,0) = f(x) + e^{-x} - 1 - \frac{x}{L}(e^{-L} - 1), \quad x \in (0,L), \\ &v_t(x,0) = 0, \quad x \in (0,L). \end{aligned}$$

Using separation of variables, we set $v(x,t) = X(x)T(t)$. This gives, with a dot indicating a time derivative and a prime a spatial derivative,

$$X\ddot{T} = TX'' - 2\gamma X\dot{T},$$

from which we get

$$\frac{\ddot{T}}{T} + 2\gamma\frac{\dot{T}}{T} = \frac{X''}{X} = \alpha,$$

where α is a constant. If $\alpha = 0$ the X equation yields $X(x) = E + Dx$, which can't satisfy the boundary conditions unless $E = D = 0$. If $\alpha > 0$, $X(x) = Ee^{\alpha x} + De^{-\alpha x}$ which again can only satisfy the boundary conditions $X(0) = X(L) = 0$ if $E = D = 0$. So we take negative α, $\alpha = -\lambda^2$, with λ real. The boundary conditions are satisfied if $\lambda = \lambda_n = \frac{n\pi}{L}$:

$$X_n = \sin\left(\frac{n\pi x}{L}\right).$$

The equation for T gives $\ddot{T} + 2\gamma\dot{T} + \frac{n^2\pi^2}{L^2}T = 0$, with characteristic roots r_n given by

$$r_n = -\gamma \pm \sqrt{\gamma^2 - \frac{n^2\pi^2}{L^2}}.$$

Assuming $\frac{\gamma L}{\pi}$ is not an integer, the roots are not repeated and we get two cases, depending on whether n is larger or smaller than $M = \frac{\gamma L}{\pi}$:

$$T_n(t) = \begin{cases} e^{-\gamma t}[A_n \sinh(\beta_n t) + B_n \cosh(\beta_n t)], & n < M, \\ e^{-\gamma t}[A_n \sin(\omega_n t) + B_n \cos(\omega_n t)], & n > M, \end{cases}$$

where $\omega_n = \sqrt{\frac{n^2\pi^2}{L^2} - \gamma^2}$ and $\beta_n = \sqrt{\gamma^2 - \frac{n^2\pi^2}{L^2}}$.
Constructing $v(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$ and $u = v - [e^{-x} - 1 - \frac{x}{L}(e^{-L} - 1)]$ we get

$$\begin{aligned} u(x,t) =& 1 - e^{-x} + \frac{x}{L}(e^{-L} - 1) + e^{-\gamma t}\sum_{n=1}^{M} \sin\left(\frac{n\pi x}{L}\right)[A_n \sinh(\beta_n t) + B_n \cosh(\beta_n t)] \\ &+ e^{-\gamma t}\sum_{n=M+1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)[A_n \sin(\omega_n t) + B_n \cos(\omega_n t)]. \end{aligned} \tag{3}$$

To find the constants we need that the initial conditions be satisfied, namely

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) B_n + 1 - e^{-x} + \frac{x}{L}(e^{-L} - 1),$$

which gives, taking B_n as the Fourier coefficients of $f(x) - 1 + e^{-x} - \frac{x}{L}(e^{-L} - 1)$,

$$B_n = \frac{2}{L}\int_0^L \left[f(x) - 1 + e^{-x} - \frac{x}{L}(e^{-L} - 1)\right] \sin\left(\frac{n\pi x}{L}\right) dx. \tag{4}$$

Similarly, we need

$$u_t(x,0) = 0 = -\gamma \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) B_n + \sum_{n=1}^{M} \sin\left(\frac{n\pi x}{L}\right) \beta_n A_n + \sum_{n=M+1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \omega_n A_n,$$

or, using $f(x) - 1 + e^{-x} - \frac{x}{L}(e^{-L} - 1) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) B_n$,

$$\gamma[f(x) - 1 + e^{-x} - \frac{x}{L}(e^{-L} - 1)] = \sum_{n=1}^{M} \sin\left(\frac{n\pi x}{L}\right) \beta_n A_n + \sum_{n=M+1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \omega_n A_n,$$

and we obtain

$$A_n = \begin{cases} \frac{\gamma}{\beta_n} B_n, & n \leq M, \\ \frac{\gamma}{\omega_n} B_n, & n > M. \end{cases} \tag{5}$$

Together, (3), (4), and (5) constitute a formal solution to the problem.