Preliminary Exam
Partial Differential Equations
9 AM - 12 PM, Fri. Jan. 11, 2019
Room: Newton Lab (ECCR 257)

Student ID:__________________________________

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points. A sheet of convenient formulae is provided.

1. **Method of characteristics.** Solve the following initial-boundary value problem:

$$\begin{aligned} u_t + \mathrm{e}^{-x} u_x &= \mathrm{e}^x, && x > 0, \quad t > 0, \\ u(x,0) &= f(x), && x > 0, \\ u(0,t) &= g(t), && t > 0. \end{aligned}$$

You will need to separate the domain into two regions, and be sure to identify the boundary between the two regions.

2. **Heat Equation.**
(a) Use energy methods to show the following initial boundary value problem has at most one solution:

$$\begin{aligned} u_t(\mathbf{x},t) &= f(t)\Delta u(\mathbf{x},t), && f(t) > 0,\ \mathbf{x} \in \Omega, \\ u(\mathbf{x},0) &= g(\mathbf{x}), && \mathbf{x} \in \Omega \\ u(\mathbf{x},t) &= h(\mathbf{x},t) && \mathbf{x} \in \partial\Omega, \qquad t > 0. \end{aligned}$$

Assume $f(t) \in C^\infty[0,\infty)$, $g(\mathbf{x}) \in C^\infty(\partial\Omega)$, and $h(\mathbf{x},t) \in C^\infty(\partial\Omega \times [0,\infty))$ are all integrable functions and the domain Ω is simply connected.
(b) Find the solution to the heat equation on the n-dimensional half space:

$$\begin{aligned} u_t(\mathbf{x},t) &= \kappa \Delta u(\mathbf{x},t), && \mathbf{x} \in \Omega \equiv \{\mathbf{x} \in \mathbb{R}^n | x_n > 0\}, \quad t > 0, \\ u(\mathbf{x},0) &= f(\mathbf{x}), && \mathbf{x} \in \Omega, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x},t) &= 0, && \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x},t) &= 0, && t > 0, \end{aligned}$$

where $\kappa > 0$ is a positive scalar, $f(\mathbf{x})$ is continuous and L^2 integrable on $\mathbb{R}^n$, and $\mathbf{n}$ is the unit normal to the boundary $\partial\Omega$.

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3. **Wave Equation.** Consider the equation

$$\begin{aligned} u_{tt}(\mathbf{x},t) &= c^2 \Delta u(\mathbf{x},t), \\ u(\mathbf{x},0) &= f(\mathbf{x}), \qquad && \mathbf{x} \in \mathbb{R}^3 \\ u_t(\mathbf{x},0) &= g(\mathbf{x}), \qquad && \mathbf{x} \in \mathbb{R}^3 \end{aligned}$$

where $c > 0$ is a positive scalar, and $f(\mathbf{x})$ and $g(\mathbf{x})$ are rapidly decaying, C^∞, and L^2 integrable functions.

(a) Find the equation the average of u:

$$\overline{u}(r,t) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(\mathbf{x},t) \sin\phi d\phi \; d\theta$$

satisfies where $\mathbf{x} = (x,y,z)$ and $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi\ \theta$, such that θ and ϕ are angles in spherical coordinates.

(b) Assume spherical symmetry of initial conditions ($f = f(r), g = g(r)$) and the solution $u = u(r,t)$, where $r \equiv |\mathbf{x}|$, and write the initial/boundary value problem for the radially symmetric function $v(r,t) = ru(r,t)$.

(c) Find the solution $v(r,t)$ and hence the solution $u(r,t)$.

4. **Green's Functions.**

(a) Consider Poisson's equation on the tilted half plane

$$\begin{aligned} \Delta u &= f(\mathbf{x}), \qquad && \mathbf{x} \in \Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 > 0\}, \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad && \mathbf{x} \in \partial\Omega. \end{aligned} \tag{1}$$

Write the associated Green's function $G_H(\mathbf{x},\mathbf{y})$ using the method of images, and verify its corresponding boundary value problem.

(b) Consider Poisson's equation on the tilted half disc:

$$\begin{aligned} \Delta u &= f(\mathbf{x}), \qquad && \mathbf{x} \in \Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 > 0 \; \& \; |\mathbf{x}| < 1\}, \\ u(\mathbf{x}) &= 0 \qquad && \mathbf{x} \in \partial\Omega. \end{aligned} \tag{2}$$

Determine the associated Green's function $G_S(\mathbf{x},\mathbf{y})$ and show it satisfies the needed boundary conditions. Then, write the solution to Eq. (2) in terms of this Green's function, and show it satisfies $u(\mathbf{x}) = 0$ on $\mathbf{x} \in \partial\Omega$.

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5. **Separation of Variables.** Consider the forced, damped wave equation

$$u_{tt} = u_{xx} - 2\gamma u_t + e^{-x}, \quad 0 \leq x \leq L, \quad t > 0,$$

with $\gamma > 0$ and boundary and initial conditions

$$\begin{aligned} &u(0,t) = u(L,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad x \in (0,L), \\ &u_t(x,0) = 0, \quad x \in (0,L). \end{aligned}$$

Using separation of variables, find a *formal* solution $u(x,t)$ to the boundary value problem in terms of the function $f(x)$. (You can assume $\frac{\gamma L}{\pi}$ is not an integer.)