Preliminary Examination (Solutions): Partial Differential Equations,
10 AM - 1 PM, Jan. 18, 2016,
Room Discovery Learning Center (DLC) Bechtel Collaboratory.

Student ID: ________________________________

There are five problems. Solve four of the five problems. Each problem is worth 25 points. A sheet of convenient formulae is provided.

1. (Solution methods) Let $\Omega = (0, \pi) \times (0, T)$, $T > 0$. Consider the initial, boundary value problem

$$
\begin{align*}
  u_t &= 2u_{xx}, \quad 0 < x < \pi, \quad t > 0 \\
  u_x(\pi, t) &= 2, \quad u(0, t) = 0, \quad t > 0, \\
  u(x, 0) &= \phi(x), \quad 0 < x < \pi.
\end{align*}
$$

(a) Find a formal solution $u(x, t)$ that solves the above initial value problem.

(b) Find sufficient conditions on $\phi$ such that the formal solution $u$ is classical, i.e., it is in $C^2(\bar{\Omega})$, functions that are twice continuously differentiable for $x \in [0, \pi]$ and continuously differentiable for $t \in [0, T]$. For full credit, you must provide a complete proof of your conclusion.

Solution:

We first need to shift the boundary condition and let $v = u - 2x$ to obtain

$$
\begin{align*}
  v(0, t) &= 0 \\
  v_x(\pi, t) &= 0 \\
  v(x, 0) &= \phi(x) - 2x.
\end{align*}
$$

We then use separation of variables $v(x, t) = X(x)T(t)$ to obtain

$$
\begin{align*}
  XT' &= 2X''T \\
  \frac{X''}{X} &= \frac{T'}{2T} = -\lambda.
\end{align*}
$$

Thus we have an equation for $X$ and an equation for $T$.

(a) Find the formal solution
i. For the $X$ equation

$$X'' = \lambda X$$

we have that $X(x) = A \cos \beta x + B \sin \beta x$ (with $\lambda = \beta^2$) and the boundary conditions give us

$$X(0) = 0 \rightarrow A = 0$$

and

$$X'(\pi) = 0 = B \beta \cos \beta \pi \rightarrow \beta_n = n + \frac{1}{2}, \quad n = 0, 1, \ldots.$$ 

These results mean

$$X_n(x) = B_n \sin((n + 1/2)x), \quad n = 0, 1, \ldots.$$ 

ii. For the $T$ equation, we have that

$$T' = -2\lambda T$$

which has solution $T(t) = Ce^{-2\lambda t}$.

iii. Therefore we obtain the formal series solution

$$v(x, t) = \sum_{n=0}^{\infty} B_n \sin(\beta_n x) e^{-2\beta_n^2 t}$$

iv. By equating this series to the initial data

$$v(x, 0) = \phi(x) - 2x = \sum_{n=0}^{\infty} B_n \sin \beta_n x$$

we have that

$$B_n = \frac{2}{\pi} \int_0^{\pi} \sin(\beta_n x) (\phi(x) - 2x) \, dx.$$

Thus the formal solution is

$$u(x, t) = \sum_{n=0}^{\infty} B_n \sin(\beta_n x) e^{-2\beta_n^2 t} + 2x. \quad (1)$$

(b) For the above formal solution to be classical, it is sufficient that $\phi(0) = 0$, $\phi'(\pi) = 2$, $\phi''(0) = 0$, and $\phi \in C^4[0, \pi]$.

**Convergence of $u(x, t)$**: To illustrate the source of these restrictions, first consider the formal series for $u(x, t)$ in Eq. (1). We require this series to be uniformly convergent for $(x, t) \in \Omega$ and the initial/boundary data to be satisfied. For uniform convergence, we invoke the Weierstrass $M$-test: $\sum_n a_n(x, t)$ is uniformly convergent if $|a_n(x, t)| < M_n$ for $(x, t) \in \bar{\Omega}$ and $\sum_n M_n < \infty$. Since
\[ |B_n \sin(\beta_n x)e^{-2\beta_n^2 t}| \leq |B_n|, \text{ we require } \sum_{n=0}^{\infty} |B_n| < \infty. \] A sufficient condition for this inequality to hold is \[ |B_n| < C_n/n^2 \text{ for } n = 1, 2, \ldots. \] We formally compute

\[
\frac{\pi}{2} B_n = \int_0^\pi (\phi(x) - 2x) \sin(\beta_n x) \, dx \\
= -\frac{1}{\beta_n} \phi(0) + \frac{1}{\beta_n} \left[ \int_0^\pi (\phi'(x) - 2) \cos(\beta_n x) \, dx \right] \\
= \frac{1}{\beta_n} \phi(\pi) + \frac{1}{\beta_n^2} \left( \phi'(x) - 2 \right) \sin(\beta_n x) \, dx \\
= \frac{1}{\beta_n} \phi(0) + \frac{1}{\beta_n^2} \phi'(x) \sin(\beta_n x) [0^\pi] - \frac{1}{\beta_n^2} \phi''(x) \sin(\beta_n x) \, dx \\
= \frac{1}{\beta_n} \phi(0) + \frac{1}{\beta_n^2} \phi'(\pi) \sin(\beta_n x) \, dx
\]

(2)

If we set \( \phi(0) = 0 \) and require \( \phi \in C^2[0, \pi] \), then we can uniformly bound the final integrand \( |\phi''(x) \sin(\beta_n x)| < M'' < \infty \) and obtain

\[ |B_n| < \frac{C}{\beta_n^2} < \frac{C}{n^2}, \quad C = \frac{2}{\pi} (|\phi'(\pi)| + 2) + 2M'', \quad n = 1, 2, \ldots, \]

as needed.

**Convergence to initial data:** Because \( \phi \in C^2[0, \pi] \) and \( \phi(0) = 0 \), \( \phi \) has a smooth, odd, periodic extension to \([-2\pi, 2\pi]\). Then the Fourier quarter-sine series for \( \phi, u(x, 0) - 2x \), is guaranteed to converge uniformly to \( \phi(x) \) for \( x \in [0, \pi] \) by Dirichlet’s theorem.

**Convergence to boundary data:** Since each term in the uniformly convergent series (1) evaluates to zero at \( x = 0 \), we have \( u(0, t) = 0 \). We check the other boundary condition next. Consider \( u_x \). In order to be able to differentiate the series (1) term-by-term, we require each differentiated term to be a) uniformly continuous and b) for the differentiated series

\[ S^{(1)}(x, t) = \sum_{n=0}^{\infty} \beta_n B_n \cos(\beta_n x)e^{-2\beta_n^2 t} \]

to be absolutely summable. If these are both true, then \( S(x, t) \) converges uniformly to the continuous function \( u_x(x, t) \) for \( (x, t) \in \overline{\Omega} \).

The first requirement is satisfied because \( \cos(\beta_n x)e^{-2\beta_n^2 t} \) is continuous for all \( (x, t) \). Since the domain \( \overline{\Omega} \) is compact, each term is uniformly continuous. \( |\beta_n B_n| < C/n^2 \) or \( |B_n| < C/n^3 \), \( n = 1, 2, \ldots \) is a sufficient condition for absolute summability by the \( M \)-test. To obtain sufficient conditions, we resume
the calculation in Eq. (2), imposing the already specified assumptions

\[ \frac{\pi}{2} B_n = \frac{1}{\beta_n^2} (\phi'(\pi) - 2) (-1)^n - \frac{1}{\beta_n^3} \int_0^\pi \phi''(x) \sin(\beta_n x) dx \]

\[ = \frac{1}{\beta_n^2} (\phi'(\pi) - 2) (-1)^n + \frac{1}{\beta_n^3} \cos(\beta_n x) \phi''(x) \bigg|_0^\pi - \frac{1}{\beta_n^3} \int_0^\pi \phi'''(x) \cos(\beta_n x) dx \]

\[ = \frac{1}{\beta_n^2} (\phi'(\pi) - 2) (-1)^n - \frac{1}{\beta_n^3} \phi''(0) - \frac{1}{\beta_n^3} \int_0^\pi \phi'''(x) \cos(\beta_n x) dx. \]

(3)

If we set \( \phi'(\pi) = 2 \) and require \( \phi \in C^4[0, \pi] \), then we can bound the final integrand \( |\phi'''(x) \cos(\beta_n x)| < M'' < \infty \) and obtain

\[ |B_n| < \frac{C}{\beta_n^3} < \frac{C}{n^3}; \quad C = \frac{2}{\pi} |\phi''(0)| + 2M'' \]

Therefore \( S^{(1)}(x, t) \) converges uniformly to \( u_x(x, t) - 2 \) for \((x, t) \in \overline{\Omega}\). We can therefore evaluate the series \( u_x(\pi, t) = S^{(1)}(\pi, t) + 2 = 2 \) as required.

**Convergence of \( u_{xx} \) and \( u_t \):** The final step is to prove uniform convergence of the series

\[ S^{(2)}(x, t) = -\sum_{n=0}^\infty \beta_n^2 B_n \sin(\beta_n x) e^{-2\beta_n^2 t}. \]

By similar arguments as given earlier, \( |B_n| < C/n^4 \), \( n = 1, 2, \ldots \) is sufficient for this. We resume the calculation (3) with all prior assumptions

\[ \frac{\pi}{2} B_n = -\frac{1}{\beta_n^3} \phi''(0) - \frac{1}{\beta_n^3} \int_0^\pi \phi'''(x) \cos(\beta_n x) dx \]

\[ = -\frac{1}{\beta_n^3} \phi''(0) - \frac{1}{\beta_n^4} \phi'''(x) \sin(\beta_n x) \bigg|_0^\pi + \frac{1}{\beta_n^4} \int_0^\pi \phi'''(x) \sin(\beta_n x) dx \]

\[ = -\frac{1}{\beta_n^3} \phi''(0) - \frac{1}{\beta_n^4} \phi'''(\pi)(-1)^n + \frac{1}{\beta_n^4} \int_0^\pi \phi'''(x) \sin(\beta_n x) dx. \]

If we set \( \phi''(0) = 0 \) and require \( \phi \in C^4[0, \pi] \), then

\[ |B_n| < \frac{C}{\beta_n^4} < \frac{C}{n^4}; \quad C = \frac{2}{\pi} |\phi'''(\pi)| + 2|M'''| \]

and \( S^{(2)}(x, t) \) converges uniformly to continuous \( u_{xx}(x, t) \) on \( \overline{\Omega} \).

Because \( u_t = 2u_{xx} \), we also have uniform convergence of \( u_t(x, t) \) to a continuous function on \( \overline{\Omega} \).

In summary, if \( \phi(0) = 0 \), \( \phi'(\pi) = 2 \), \( \phi''(0) = 0 \), and \( \phi \in C^4[0, \pi] \) then \( u \) in Eq. (1) is \( C^2(\overline{\Omega}) \) and is a classical solution to the initial, boundary value problem.
2. **(Heat equation)** Consider the following initial-boundary value problem for the heat equation

\[
\begin{align*}
  u_t &= u_{xx}, & x \in (0, 1), & t > 0, \\
  u(x, 0) &= x(1 - x), & x \in (0, 1), \\
  u(0, t) &= u(1, t) = 0, & t > 0.
\end{align*}
\]

Assume the existence of a classical solution \( u(x, t) \).

(a) Prove the uniqueness of this solution.

(b) Show that \( u(x, t) > 0 \) on \( x \in (0, 1) \) and \( t > 0 \).

(c) For each \( t > 0 \), let \( \mu(t) := \max_{x \in [0,1]} u(x, t) \). Show that \( \mu(t) \) is a nonincreasing function of \( t \).

**Solution:**

(a) Assume two solutions \( u \) and \( v \), then \( w = u - v \) solves the problem

\[
\begin{align*}
  w_t &= w_{xx}, & x \in (0, 1), & t > 0, \\
  w(x, 0) &= 0, & x \in (0, 1), \\
  w(0, t) &= w(1, t) = 0, & t > 0.
\end{align*}
\]

The maximum principle ensures \( \max_R w(x, t) = 0 \) and the minimum principle ensures \( \min_R w(x, t) = 0 \) for any rectangle \( R = [0, 1] \times [0, T] \). Thus \( w \equiv 0 \) on any \( R \), so \( u \equiv v \).

(b) By the minimum principle, we know \( \min_R u(x, t) = 0 \) for any rectangle \( R = [0, 1] \times [0, T] \), since \( u(0, t) = u(1, t) = 0 < x(1 - x) \) for \( x \in (0, 1) \). By the strong minimum principle, we know that if the minimum \( \min_R u(x, t) = 0 \) is also obtained at a point \( (x^*, t^*) \) for \( x^* \in (0, 1) \) and \( t \in (0, T) \), then \( u \equiv 0 \) on \( R \). However, we know this cannot be since \( u(x, 0) = x(1 - x) \), so \( u(x, t) \) must only obtain its minimum on \( x = 0 \) and \( x = 1 \).

(c) By the maximum principle, \( u(1/2, 0) = 1/4 = \mu(0) \). At each \( t > 0 \), define \( X(t) \) such that \( \mu(t) = u(X(t), t) \). Differentiating, we find

\[
\mu'(t) = u_x(X(t), t)X'(t) + u_t(X(t), t).
\]

At \( (X(t), t) \), \( u_x(X(t), t) = 0 \) and \( u_{xx}(X(t), t) \leq 0 \). Thus, \( \mu'(t) = u_{xx}(X(t), t) \leq 0 \), so \( \mu(t) \) is nonincreasing.

Alternatively, one could define for each \( t_0 > 0 \) an initial boundary value problem with \( u(x, t_0) \) as the initial condition, so that the maximum principle ensures the maximum on \( [0, 1] \times [t_0, T] \) lies at \( t_0 \) for any \( T > t_0 > 0 \). Thus, \( u(x, t) \leq u(x, t_0) \) for any \( t > t_0 > 0 \), so \( \mu(t) \) is nonincreasing.
3. **(Green’s function)** Consider the boundary value problem

\[- \Delta u(x) = f(x), \quad x \in \Omega \subset \mathbb{R}^3, \]
\[u(x) = g(x), \quad x \in \partial \Omega. \quad (4)\]

(a) Formulate a boundary value problem for Green’s function \(G(x, y) = \Phi(x - y) - \phi^x(y)\) for \(x \in \Omega\) using the fundamental solution \(\Phi(x) = (4\pi|\mathbf{x}|)^{-1}\).

(b) Prove that Green’s function, if it exists, is unique.

(c) Construct Green’s function when \(\Omega = B(0, 1) \cap \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\right\}\), where \(B(0, 1)\) is the unit sphere.

**Solution:**

(a) Fix \(x \in \Omega\) and let \(G(x, y) = \Phi(x - y) - \phi^x(y)\). Then \(\phi^x\) satisfies

\[\Delta_y \phi^x(y) = 0, \quad y \in \Omega, \]
\[\phi^x(y) = \Phi(x - y), \quad y \in \partial \Omega.\]

(b) Fix \(x \in \Omega\) and assume the existence of two Green’s functions \(G_j(x, y) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} + \phi_j^x(y)\) for \(j = 1, 2\) where each \(\phi_j^x\) satisfies

\[\Delta_y \phi_j^x(y) = 0, \quad y \in \Omega, \]
\[\phi_j^x(y) = \Phi(x - y), \quad y \in \partial \Omega, \quad j = 1, 2.\]

Then \(u(y) = G_1(x, y) - G_2(x, y)\) is independent of \(x\) and satisfies

\[\Delta_y u(y) = 0, \quad y \in \Omega, \]
\[u(y) = 0, \quad y \in \partial \Omega, \]

i.e., \(u\) is harmonic in \(\Omega\). By the maximum/minimum principles and the boundary data, \(u(y) \equiv 0\) so that \(G_1(x, y) = G_2(x, y)\) and Green’s function is unique.

(c) Fix \(x \in \Omega\) and let \(G(x, y) = \Phi(x - y) - \phi^x(y)\). First, we construct Green’s function for the unit sphere \(B(0, 1)\) with the method of images. Let

\[\phi_B^x(y) = \Phi(|\mathbf{x}|(\bar{x} - y)), \quad \bar{x} = \frac{x}{|x|^2}.\]

Since \(\bar{x}\) is not in \(B(0, 1)\), \(\phi_B^x(y)\) is harmonic in \(B(0, 1)\). Let \(|\mathbf{y}| = 1\), then we compute

\[||\mathbf{x}|(\bar{x} - \mathbf{y})|^2 = \left|\frac{x}{|x|} - |\mathbf{x}|\mathbf{y}\right|^2 \]
\[= 1 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{x}|^2 \]
\[= |\mathbf{x} - \mathbf{y}|^2.\]
Then, for \( y \in \partial B(0,1) \), we have
\[
\phi_B^x(y) = \Phi(|x|(|\tilde{x} - y|)) \\
= \Phi(x - y),
\]
as required and Green’s function for the unit sphere is \( G_B(x, y) = \Phi(x - y) - \phi_B^x(y) \). We now use the method of images again to obtain Green’s function for the upper hemisphere \( \Omega \) by reflecting about the plane \( x_3 = 0 \). Let
\[
G(x, y) = G_B(x, y) - G_B(\tilde{x}, y), \quad \tilde{x} = (x_1, x_2, -x_3)
\]
\[
= \Phi(x - y) - \Phi(|x|(|\tilde{x} - y|)) - \Phi(\tilde{x} - y) + \Phi(|\tilde{x}|(|\tilde{x} - y|)) \\
= \Phi(x - y) - \phi^x(y).
\]
Since \( \tilde{x}, \hat{x}, \) and \( \tilde{\hat{x}} \) are not in \( \Omega \), \( \phi^x(y) \) is harmonic for \( y \in \Omega \). We need to show that \( \phi^x(y) = \Phi(x - y) \) for \( y \in \partial \Omega \). If \( |y| = 1 \) then as before, \( |x||\tilde{x} - y| = |x - y| \). Similarly, \( |\tilde{x}||\tilde{x} - y| = |\tilde{x} - y| \). Then, for \( |y| = 1 \) and \( y_3 \geq 0 \),
\[
\phi^x(y) = \Phi(|x|(|\tilde{x} - y|)) + \Phi(\tilde{x} - y) - \Phi(|\tilde{x}|(|\tilde{x} - y|)) \\
= \Phi(x - y) + \Phi(\tilde{x} - y) - \Phi(\tilde{x} - y) \\
= \Phi(x - y),
\]
as required. Finally, if \( y = (y_1, y_2, 0) \), \( y_1^2 + y_2^2 \leq 1 \), then \( |x||\tilde{x} - y| = |\tilde{x}||\tilde{x} - y| \) and \( |\tilde{x} - y| = |x - y| \), so that
\[
\phi^x(y) = \Phi(|x|(|\tilde{x} - y|)) + \Phi(\tilde{x} - y) - \Phi(|\tilde{x}|(|\tilde{x} - y|)) \\
= \Phi(|\tilde{x}|(|\tilde{x} - y|)) + \Phi(x - y) - \Phi(\tilde{x} - y) \\
= \Phi(x - y),
\]
and the construction is complete.
4. **(Wave equation)** Consider the wave equation

\[
\begin{aligned}
&u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \\
&u(x, 0) = \phi(x), \quad x \in \mathbb{R}, \\
&u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}.
\end{aligned}
\] (5)

Assume the existence of a classical solution \(u(x, t)\).

(a) If \(\phi(x)\) and \(\psi(x)\) are both odd functions of \(x\), show that the solution \(u(x, t)\) is odd in \(x \in \mathbb{R}\) for \(t > 0\).

(b) Find a solution to Eq. (5) assuming \(\phi(x) \equiv 0\) and \(\psi(x) \equiv 0\), and prove that it is unique.

(c) Assume \(\phi(x)\) and \(\psi(x)\) have compact support, fix \(c = 1\), and define the kinetic \(K(t) = \frac{1}{2} \int_{\mathbb{R}} u_t(x, t)^2 dx\) and potential \(P(t) = \frac{1}{2} \int_{\mathbb{R}} u_x(x, t)^2 dx\) energies. Show that \(K(t) = \overline{P}(t)\) for all \(t\) sufficiently large.

**Solution:**

(a) By d’Alembert’s formula

\[
\begin{aligned}
u(-x, t) &= \frac{1}{2} [\phi(-x + ct) + \phi(-x - ct)] + \frac{1}{2c} \int_{-x-ct}^{-x+ct} \psi(s) ds \\
&= \frac{1}{2} [-\phi(x - ct) - \phi(x + ct)] - \frac{1}{2c} \int_{x+ct}^{x-ct} \psi(s) ds \\
&= - \left( \frac{1}{2} [\phi(x - ct) + \phi(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds \right) = -u(x, t)
\end{aligned}
\]

(b) \(u \equiv 0\) is a solution since \(u(x, 0) \equiv 0 \equiv \phi(x)\); \(u_t(x, 0) \equiv 0 \equiv \psi(x)\); \(u_{tt} = 0 = u_{xx}\). Define the energy \(E(t) = \frac{1}{2} \int_{\mathbb{R}} u_t(x, t)^2 + c^2 u_x(x, t)^2 dx = \overline{E}\), constant due to conservation. Thus, since \(E(0) = \frac{1}{2} \int_{\mathbb{R}} \phi(x)^2 + c^2 \psi(x)^2 dx = 0 = \overline{E}\). Since \(u_t^2, u_x^2 \geq 0\), this implies \(u_t \equiv u_x \equiv 0\), and since \(u(x, 0) \equiv 0\), then \(u(x, t) \equiv 0\).

(c) By d’Alembert’s formula

\[
\begin{aligned}
u(x, t) &= \frac{1}{2} [\phi(x + t) + \phi(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) ds \\
&= \frac{1}{2} [\phi'(x + t) - \phi'(x - t)] + \frac{1}{2} [\psi(x + t) + \psi(x - t)]
\end{aligned}
\]

so

\[
u_t(x, t) = \frac{1}{2} [\phi'(x + t) - \phi'(x - t)] + \frac{1}{2} [\psi(x + t) + \psi(x - t)]
\]

and

\[
u_x(x, t) = \frac{1}{2} [\phi'(x + t) + \phi'(x - t)] + \frac{1}{2} [\psi(x + t) - \psi(x - t)].
\]
Now, since both $\phi$ and $\psi$ have compact support, for sufficiently large $t$, all terms

\[
0 = \int_{\mathbb{R}} \phi'(x+t)\phi'(x-t)dx
\]

\[
\equiv \int_{\mathbb{R}} \phi'(x+t)\psi(x-t)dx
\]

\[
\equiv \int_{\mathbb{R}} \phi'(x-t)\psi(x+t)dx
\]

\[
\equiv \int_{\mathbb{R}} \psi(x-t)\psi(x+t)dx.
\]

That is, if $\phi(x)$ is supported on $x \in [a, b]$ and $\psi(x)$ is supported on $x \in [m, n]$, then once $t > \max(n - m, b - a, n - a, b - m)/2 = t_0$, the length $2t$ will be wider than the region containing the support of both functions, so $\phi'(x)$ and $\psi(x)$ must be zero either at $x + t$ or $x - t$. Thus, we compute

\[
K(t) = \frac{1}{8} \int_{\mathbb{R}} \left[ \phi'(x+t)^2 + \phi'(x-t)^2 + 2\phi'(x+t)\psi(x+t) - 2\phi'(x-t)\psi(x-t) + \psi(x+t)^2 + \psi(x-t)^2 \right]dx.
\]

For any finite $t > t_0$, we can change variables of all terms $z = x + t$ and $z = x - t$ and not change the integral, so

\[
K(t) = \frac{1}{4} \int_{\mathbb{R}} \left[ \phi'(z)^2 + \psi(z)^2 \right]dz.
\]

Similarly,

\[
P(t) = \frac{1}{8} \int_{\mathbb{R}} \left[ \phi'(x+t)^2 + \phi'(x-t)^2 + 2\phi'(x+t)\psi(x+t) - 2\phi'(x-t)\psi(x-t) + \psi(x+t)^2 + \psi(x-t)^2 \right]dx.
\]

and changing variables,

\[
P(t) = \frac{1}{4} \int_{\mathbb{R}} \left[ \phi'(z)^2 + \psi(z)^2 \right]dz,
\]

so $K(t) = P(t)$ for $t > t_0$. 

5. **(Method of characteristics)** Consider the quasilinear equation

\[(y + u)u_x + yu_y = x - y.\]

(a) Give an example of a connected curve \(\Gamma \subset \mathbb{R}^2\) such that the Cauchy problem with prescribed data on that curve cannot be solved.

(b) Given the Cauchy data \(u(x,1) = 1 + x\). What are the characteristic curves? Find the solution. For what values of \((x,y) \in \mathbb{R}^2\) does the solution exist?

**Solution:**

(a) We require the data \(u(x_0(s), y_0(s)) = u_0(s), s \in I\) to lie on a characteristic, i.e., for the vector field \(v = (y + u, y)\) to be tangent to the initial curve. For example, \(v\) is tangent to the curve \(\Gamma = \{(x,0) \mid x \in \mathbb{R}\}\) for any choice of initial data. Note that the zero Jacobian condition

\[
J = \begin{vmatrix} x'_0(s) & y'_0(s) \\ y_0(s) + u_0(s) & y_0(s) \end{vmatrix} = x'_0y_0 - y'_0(y_0 + u_0) = 0 \quad \text{for all} \quad s \in I,
\]

is necessary but not sufficient to prove that the problem cannot be solved.

(b) We parametrize the data as \(x = s, y = 1, u = 1 + s\) for \(s \in \mathbb{R}\). Then the characteristic equations for \(x(\xi, s), y(\xi, s), \) and \(u(\xi, s)\) are

\[
\begin{align*}
x_\xi &= y + u, \quad x(0, s) = s, \\
y_\xi &= y, \quad y(0, s) = 1, \\
u_\xi &= x - y, \quad u(0, s) = 1 + s, \quad s \in \mathbb{R}.
\end{align*}
\]

Taking another derivative of the first equation, we obtain

\[
x_{\xi \xi} = y_\xi + u_\xi \\
= y + x - y \\
= x,
\]

which has solution \(x = A(s)e^\xi + B(s)e^{-\xi}\). The initial data imply \(x(0, s) = A(s) + B(s) = s\) and \(x_\xi(0, s) = A(s) - B(s) = y(0, s) + u(0, s) = 2 + s\). Then \(A(s) = 1 + s\) and \(B(s) = -1\) so that

\[
x(\xi, s) = (1 + s)e^\xi - e^{-\xi}.
\]

The equation for \(y\) is solved

\[
y(\xi, s) = e^\xi.
\]

Then the characteristic curves are

\[
x = (1 + s)y - \frac{1}{y}
\]
for each $s \in \mathbb{R}$. We solve the equation for $u$
\[
    u_{\xi} = se^{\xi} - e^{-\xi} \quad \Rightarrow \quad u(\xi, s) = se^{\xi} + e^{-\xi}.
\]
Undoing the characteristic transformation, we obtain the solution
\[
    u(x, y) = x - y + \frac{2}{y}, \quad x \in \mathbb{R}, \quad y \neq 0,
\]
which exists for $y \neq 0$. 