Preliminary Examination (Solutions): Partial Differential Equations,
10 AM - 1 PM, Jan. 18, 2016,
Room Discovery Learning Center (DLC) Bechtel Collaboratory.

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

Student ID:________________________________

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points. A sheet of convenient formulae is provided.

1. **(Solution methods)** Let $\Omega = (0,\pi) \times (0,T)$, $T > 0$. Consider the initial, boundary value problem

$$\begin{aligned} u_t = 2u_{xx}, \quad 0 < x < \pi, \quad t > 0 \\ u_x(\pi,t) = 2, \quad u(0,t) = 0, \qquad t > 0, \\ u(x,0) = \phi(x), \quad 0 < x < \pi. \end{aligned}$$

(a) Find a formal solution $u(x,t)$ that *solves* the above initial value problem.

(b) Find sufficient conditions on ϕ such that the formal solution u is classical, i.e., it is in $\mathcal{C}_1^2(\bar{\Omega})$, functions that are twice continuously differentiable for $x \in [0,\pi]$ and continuously differentiable for $t \in [0,T]$. For full credit, you must provide a complete proof of your conclusion.

Solution:

We first need to shift the boundary condition and let $v = u - 2x$ to obtain

$$\begin{aligned} v(0,t) &= 0 \\ v_x(\pi,t) &= 0 \\ v(x,0) &= \phi(x) - 2x\,. \end{aligned}$$

We then use separation of variables $v(x,t) = X(x)T(t)$ to obtain

$$\begin{aligned} XT' &= 2X''T \\ \frac{X''}{X} = \frac{T'}{2T} &= -\lambda\,. \end{aligned}$$

Thus we have an equation for X and an equation for T.

(a) Find the formal solution

i. For the X equation

$$X'' = \lambda X$$

we have that $X(x) = A\cos\beta x + B\sin\beta x$ (with $\lambda = \beta^2$) and the boundary conditions give us

$$X(0) = 0 \rightarrow A = 0$$

and

$$X'(\pi) = 0 = B\beta\cos\beta\pi \rightarrow \beta_n = n + \frac{1}{2}, \quad n = 0, 1, \dots .$$

These results mean

$$X_n(x) = B_n \sin((n + 1/2)x), \quad n = 0, 1, \dots .$$

ii. For the T equation, we have that

$$T' = -2\lambda T$$

which has solution $T(t) = Ce^{-2\lambda t}$.

iii. Therefore we obtain the formal series solution

$$v(x,t) = \sum_{n=0}^{\infty} B_n \sin(\beta_n x) e^{-2\beta_n^2 t}$$

iv. By equating this series to the initial data

$$v(x,0) = \phi(x) - 2x = \sum_{n=0}^{\infty} B_n \sin\beta_n x$$

we have that

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin(\beta_n x)\,(\phi(x) - 2x)\,\mathrm{d}x.$$

Thus the formal solution is

$$u(x,t) = \sum_{n=0}^{\infty} B_n \sin(\beta_n x) e^{-2\beta_n^2 t} + 2x\,. \tag{1}$$

(b) For the above formal solution to be classical, it is sufficient that $\phi(0) = 0$, $\phi'(\pi) = 2$, $\phi''(0) = 0$, and $\phi \in C^4[0,\pi]$.

Convergence of $u(x,t)$: To illustrate the source of these restrictions, first consider the formal series for $u(x,t)$ in Eq. (1). We require this series to be uniformly convergent for $(x,t) \in \overline{\Omega}$ and the initial/boundary data to be satisfied. For uniform convergence, we invoke the Weierstrass M-test: $\sum_n a_n(x,t)$ is uniformly convergent if $|a_n(x,t)| < M_n$ for $(x,t) \in \overline{\Omega}$ and $\sum_n M_n < \infty$. Since

$|B_n \sin(\beta_n x)e^{-2\beta_n^2 t}| \le |B_n|$, we require $\sum_{n=0}^{\infty} |B_n| < \infty$. A sufficient condition for this inequality to hold is $|B_n| < C_n/n^2$ for $n = 1, 2, \ldots$. We formally compute

$$\begin{aligned}
\frac{\pi}{2} B_n &= \int_0^\pi (\phi(x) - 2x)\sin(\beta_n x)\mathrm{d}x \\
&= -\frac{1}{\beta_n}(\phi(x) - 2x)\cos(\beta_n x)\Big|_0^\pi + \frac{1}{\beta_n}\int_0^\pi (\phi'(x) - 2)\cos(\beta_n x)\mathrm{d}x \\
&= \frac{1}{\beta_n}\phi(0) + \frac{1}{\beta_n}\int_0^\pi (\phi'(x) - 2)\cos(\beta_n x)\mathrm{d}x \qquad (2) \\
&= \frac{1}{\beta_n}\phi(0) + \frac{1}{\beta_n^2}(\phi'(x) - 2)\sin(\beta_n x)\Big|_0^\pi - \frac{1}{\beta_n^2}\int_0^\pi \phi''(x)\sin(\beta_n x)\mathrm{d}x \\
&= \frac{1}{\beta_n}\phi(0) + \frac{1}{\beta_n^2}(\phi'(\pi) - 2)(-1)^n - \frac{1}{\beta_n^2}\int_0^\pi \phi''(x)\sin(\beta_n x)\mathrm{d}x
\end{aligned}$$

If we set $\phi(0) = 0$ and require $\phi \in C^2[0, \pi]$, then we can uniformly bound the final integrand $|\phi''(x)\sin(\beta_n x)| < M'' < \infty$ and obtain

$$|B_n| < \frac{C}{\beta_n^2} < \frac{C}{n^2}, \quad C = \frac{2}{\pi}(|\phi'(\pi)| + 2) + 2M'', \quad n = 1, 2, \ldots,$$

as needed.

Convergence to initial data: Because $\phi \in C^2[0, \pi]$ and $\phi(0) = 0$, ϕ has a smooth, odd, periodic extension to $[-2\pi, 2\pi]$. Then the Fourier quarter-sine series for ϕ, $u(x, 0) - 2x$, is guaranteed to converge uniformly to $\phi(x)$ for $x \in [0, \pi]$ by Dirichlet's theorem.

Convergence to boundary data: Since each term in the uniformly convergent series (1) evaluates to zero at $x = 0$, we have $u(0, t) = 0$. We check the other boundary condition next. Consider u_x. In order to be able to differentiate the series (1) term-by-term, we require each differentiated term to be a) uniformly continuous and b) for the differentiated series

$$S^{(1)}(x, t) = \sum_{n=0}^{\infty} \beta_n B_n \cos(\beta_n x)e^{-2\beta_n^2 t}$$

to be absolutely summable. If these are both true, then $S(x, t)$ converges uniformly to the continuous function $u_x(x, t)$ for $(x, t) \in \overline{\Omega}$.

The first requirement is satisfied because $\cos(\beta_n x)e^{-2\beta_n^2 t}$ is continuous for all (x, t). Since the domain $\overline{\Omega}$ is compact, each term is uniformly continuous. $|\beta_n B_n| < C/n^2$ or $|B_n| < C/n^3$, $n = 1, 2, \ldots$ is a sufficient condition for absolute summability by the M-test. To obtain sufficient conditions, we resume the calculation in Eq. (2), imposing the already specified assumptions

$$\begin{aligned}\frac{\pi}{2}B_n &= \frac{1}{\beta_n^2}(\phi'(\pi)-2)(-1)^n - \frac{1}{\beta_n^2}\int_0^\pi \phi''(x)\sin(\beta_n x)\mathrm{d}x \\ &= \frac{1}{\beta_n^2}(\phi'(\pi)-2)(-1)^n + \frac{1}{\beta_n^3}\cos(\beta_n x)\phi''(x)\Big|_0^\pi - \frac{1}{\beta_n^3}\int_0^\pi \phi'''(x)\cos(\beta_n x)\mathrm{d}x \\ &= \frac{1}{\beta_n^2}(\phi'(\pi)-2)(-1)^n - \frac{1}{\beta_n^3}\phi''(0) - \frac{1}{\beta_n^3}\int_0^\pi \phi'''(x)\cos(\beta_n x)\mathrm{d}x.\end{aligned} \tag{3}$$

If we set $\phi'(\pi) = 2$ and require $\phi \in C^3[0,\pi]$, then we can bound the final integrand $|\phi'''(x)\cos(\beta_n x)| < M''' < \infty$ and obtain

$$|B_n| < \frac{C}{\beta_n^3} < \frac{C}{n^3}, \quad C = \frac{2}{\pi}|\phi''(0)| + 2M''', \quad n = 1, 2, \ldots.$$

Therefore $S^{(1)}(x,t)$ converges uniformly to $u_x(x,t) - 2$ for $(x,t) \in \overline{\Omega}$. We can therefore evaluate the series $u_x(\pi,t) = S^{(1)}(\pi,t) + 2 = 2$ as required.

Convergence of u_{xx} and u_t: The final step is to prove uniform convergence of the series

$$S^{(2)}(x,t) = -\sum_{n=0}^{\infty} \beta_n^2 B_n \sin(\beta_n x) e^{-2\beta_n^2 t}.$$

By similar arguments as given earlier, $|B_n| < C/n^4$, $n = 1, 2, \ldots$ is sufficient for this. We resume the calculation (3) with all prior assumptions

$$\begin{aligned}\frac{\pi}{2}B_n &= -\frac{1}{\beta_n^3}\phi''(0) - \frac{1}{\beta_n^3}\int_0^\pi \phi'''(x)\cos(\beta_n x)\mathrm{d}x \\ &= -\frac{1}{\beta_n^3}\phi''(0) - \frac{1}{\beta_n^4}\phi'''(x)\sin(\beta_n x)\Big|_0^\pi + \frac{1}{\beta_n^4}\int_0^\pi \phi''''(x)\sin(\beta_n x)\mathrm{d}x \\ &= -\frac{1}{\beta_n^3}\phi''(0) - \frac{1}{\beta_n^4}\phi'''(\pi)(-1)^n + \frac{1}{\beta_n^4}\int_0^\pi \phi''''(x)\sin(\beta_n x)\mathrm{d}x.\end{aligned}$$

If we set $\phi''(0) = 0$ and require $\phi \in C^4[0,\pi]$, then

$$|B_n| < \frac{C}{\beta_n^4} < \frac{C}{n^4}, \quad C = \frac{2}{\pi}|\phi'''(\pi)| + 2|M''''|, \quad n = 1, 2, \ldots,$$

and $S^{(2)}(x,t)$ converges uniformly to continuous $u_{xx}(x,t)$ on $\overline{\Omega}$.
Because $u_t = 2u_{xx}$, we also have uniform convergence of $u_t(x,t)$ to a continuous function on $\overline{\Omega}$.

In summary, if $\phi(0) = 0$, $\phi'(\pi) = 2$, $\phi''(0) = 0$, and $\phi \in C^4[0,\pi]$ then u in Eq. (1) is $C_1^2(\overline{\Omega})$ and is a classical solution to the initial, boundary value problem.

2. **(Heat equation)** Consider the following initial-boundary value problem for the heat equation

$$\begin{cases} u_t = u_{xx}, & x \in (0,1), \quad t > 0, \\ u(x,0) = x(1-x), & x \in (0,1), \\ u(0,t) = u(1,t) = 0, & t > 0. \end{cases}$$

Assume the existence of a classical solution $u(x,t)$.

(a) Prove the uniqueness of this solution.

(b) Show that $u(x,t) > 0$ on $x \in (0,1)$ and $t > 0$.

(c) For each $t > 0$, let $\mu(t) := \max_{x \in [0,1]} u(x,t)$. Show that $\mu(t)$ is a nonincreasing function of t.

Solution:

(a) Assume two solutions u and v, then $w = u - v$ solves the problem

$$\begin{cases} w_t = w_{xx}, & x \in (0,1), \quad t > 0, \\ w(x,0) = 0, & x \in (0,1), \\ w(0,t) = w(1,t) = 0, & t > 0. \end{cases}$$

The maximum principle ensures $\max_R w(x,t) = 0$ and the minimum principle ensures $\min_R w(x,t) = 0$ for any rectangle $R = [0,1] \times [0,T]$. Thus $w \equiv 0$ on any R, so $u \equiv v$.

(b) By the minimum principle, we know $\min_R u(x,t) = 0$ for any rectangle $R = [0,1] \times [0,T]$, since $u(0,t) = u(1,t) = 0 < x(1-x)$ for $x \in (0,1)$. By the strong minimum principle, we know that if the minimum $\min_R u(x,t) = 0$ is also obtained at a point (x^*, t^*) for $x^* \in (0,1)$ and $t \in (0,T)$, then $u \equiv 0$ on R. However, we know this cannot be since $u(x,0) = x(1-x)$, so $u(x,t)$ must only obtain its minimum on $x = 0$ and $x = 1$.

(c) By the maximum principle, $u(1/2, 0) = 1/4 = \mu(0)$. At each $t > 0$, define $X(t)$ such that $\mu(t) = u(X(t), t)$. Differentiating, we find

$$\mu'(t) = u_x(X(t),t)X'(t) + u_t(X(t),t).$$

At $(X(t),t)$, $u_x(X(t),t) = 0$ and $u_{xx}(X(t),t) \leq 0$. Thus, $\mu'(t) = u_{xx}(X(t),t) \leq 0$, so $\mu(t)$ is nonincreasing.

Alternatively, one could define for each $t_0 > 0$ an initial boundary value problem with $u(x,t_0)$ as the initial condition, so that the maximum principle ensures the maximum on $[0,1] \times [t_0, T]$ lies at t_0 for any $T > t_0 > 0$. Thus, $u(x,t) \leq u(x,t_0)$ for any $t > t_0 > 0$, so $\mu(t)$ is nonincreasing.

3. **(Green's function)** Consider the boundary value problem

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^3, \\ u(\mathbf{x}) &= g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \end{aligned} \tag{4}$$

(a) Formulate a boundary value problem for Green's function $G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \phi^x(\mathbf{y})$ for $\mathbf{x} \in \Omega$ using the fundamental solution $\Phi(\mathbf{x}) = (4\pi|\mathbf{x}|)^{-1}$.

(b) Prove that Green's function, if it exists, is unique.

(c) Construct Green's function when

$$\Omega = B(0,1) \cap \left\{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0 \right\},$$

where $B(0,1)$ is the unit sphere.

Solution:

(a) Fix $\mathbf{x} \in \Omega$ and let $G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \phi^x(\mathbf{y})$. Then ϕ^x satisfies

$$\begin{aligned} \Delta_y \phi^x(\mathbf{y}) &= 0, \quad \mathbf{y} \in \Omega, \\ \phi^x(\mathbf{y}) &= \Phi(\mathbf{x} - \mathbf{y}), \quad \mathbf{y} \in \partial\Omega. \end{aligned}$$

(b) Fix $\mathbf{x} \in \Omega$ and assume the existence of two Green's functions $G_j(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} + \phi_j^x(\mathbf{y})$ for $j = 1, 2$ where each ϕ_j^x satisfies

$$\begin{aligned} \Delta_y \phi_j^x(\mathbf{y}) &= 0, \quad \mathbf{y} \in \Omega, \\ \phi_j^x(\mathbf{y}) &= \Phi(\mathbf{x} - \mathbf{y}), \quad \mathbf{y} \in \partial\Omega, \quad j = 1, 2. \end{aligned}$$

Then $u(\mathbf{y}) = G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$ is independent of $\mathbf{x}$ and satisfies

$$\begin{aligned} \Delta_y u(\mathbf{y}) &= 0, \quad \mathbf{y} \in \Omega, \\ u(\mathbf{y}) &= 0, \quad \mathbf{y} \in \partial\Omega, \end{aligned}$$

i.e., u is harmonic in Ω. By the maximum/minimum principles and the boundary data, $u(\mathbf{y}) \equiv 0$ so that $G_1(\mathbf{x}, \mathbf{y}) = G_2(\mathbf{x}, \mathbf{y})$ and Green's function is unique.

(c) Fix $\mathbf{x} \in \Omega$ and let $G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \phi^x(\mathbf{y})$. First, we construct Green's function for the unit sphere $B(0,1)$ with the method of images. Let

$$\phi_B^x(\mathbf{y}) = \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})), \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2}.$$

Since $\tilde{\mathbf{x}}$ is not in $B(0,1)$, $\phi_B^x(\mathbf{y})$ is harmonic in $B(0,1)$. Let $|\mathbf{y}| = 1$, then we compute

$$\begin{aligned} ||\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})|^2 &= \left| \frac{\mathbf{x}}{|\mathbf{x}|} - |\mathbf{x}|\mathbf{y} \right|^2 \\ &= 1 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{x}|^2 \\ &= |\mathbf{x} - \mathbf{y}|^2. \end{aligned}$$

Then, for $\mathbf{y} \in \partial B(0,1)$, we have

$$\begin{aligned} \phi_B^x(\mathbf{y}) &= \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) \\ &= \Phi(\mathbf{x} - \mathbf{y}), \end{aligned}$$

as required and Green's function for the unit sphere is $G_B(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \phi_B^x(\mathbf{y})$. We now use the method of images again to obtain Green's function for the upper hemisphere Ω by reflecting about the plane $x_3 = 0$. Let

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}) &= G_B(\mathbf{x}, \mathbf{y}) - G_B(\hat{\mathbf{x}}, \mathbf{y}), \quad \hat{\mathbf{x}} = (x_1, x_2, -x_3) \\ &= \Phi(\mathbf{x} - \mathbf{y}) - \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) - \Phi(\hat{\mathbf{x}} - \mathbf{y}) + \Phi(|\hat{\mathbf{x}}|(\tilde{\hat{\mathbf{x}}} - \mathbf{y})) \\ &= \Phi(\mathbf{x} - \mathbf{y}) - \phi^x(\mathbf{y}). \end{aligned}$$

Since $\tilde{\mathbf{x}}$, $\hat{\mathbf{x}}$, and $\tilde{\hat{\mathbf{x}}}$ are not in Ω, $\phi^x(\mathbf{y})$ is harmonic for $\mathbf{y} \in \Omega$. We need to show that $\phi^x(\mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y})$ for $\mathbf{y} \in \partial\Omega$. If $|\mathbf{y}| = 1$ then as before, $|\mathbf{x}||\tilde{\mathbf{x}} - \mathbf{y}| = |\mathbf{x} - \mathbf{y}|$. Similarly, $|\hat{\mathbf{x}}||\tilde{\hat{\mathbf{x}}} - \mathbf{y}| = |\hat{\mathbf{x}} - \mathbf{y}|$. Then, for $|\mathbf{y}| = 1$ and $y_3 \geq 0$,

$$\begin{aligned} \phi^x(\mathbf{y}) &= \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) + \Phi(\hat{\mathbf{x}} - \mathbf{y}) - \Phi(|\hat{\mathbf{x}}|(\tilde{\hat{\mathbf{x}}} - \mathbf{y})) \\ &= \Phi(\mathbf{x} - \mathbf{y}) + \Phi(\hat{\mathbf{x}} - \mathbf{y}) - \Phi(\hat{\mathbf{x}} - \mathbf{y}) \\ &= \Phi(\mathbf{x} - \mathbf{y}), \end{aligned}$$

as required. Finally, if $\mathbf{y} = (y_1, y_2, 0)$, $y_1^2 + y_2^2 \leq 1$, then $|\mathbf{x}||\tilde{\mathbf{x}} - \mathbf{y}| = |\hat{\mathbf{x}}||\tilde{\hat{\mathbf{x}}} - \mathbf{y}|$ and $|\hat{\mathbf{x}} - \mathbf{y}| = |\mathbf{x} - \mathbf{y}|$ so that

$$\begin{aligned} \phi^x(\mathbf{y}) &= \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) + \Phi(\hat{\mathbf{x}} - \mathbf{y}) - \Phi(|\hat{\mathbf{x}}|(\tilde{\hat{\mathbf{x}}} - \mathbf{y})) \\ &= \Phi(|\hat{\mathbf{x}}|(\tilde{\hat{\mathbf{x}}} - \mathbf{y})) + \Phi(\mathbf{x} - \mathbf{y}) - \Phi(\hat{\mathbf{x}} - \mathbf{y}) \\ &= \Phi(\mathbf{x} - \mathbf{y}), \end{aligned}$$

and the construction is complete.

4. **(Wave equation)** Consider the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

Assume the existence of a classical solution $u(x,t)$.

(a) If $\phi(x)$ and $\psi(x)$ are both odd functions of x, show that the solution $u(x,t)$ is odd in $x \in \mathbb{R}$ for $t > 0$.

(b) Find a solution to Eq. (5) assuming $\phi(x) \equiv 0$ and $\psi(x) \equiv 0$, and prove that it is unique.

(c) Assume $\phi(x)$ and $\psi(x)$ have compact support, fix $c = 1$, and define the kinetic $K(t) = \frac{1}{2}\int_{\mathbb{R}} u_t(x,t)^2 \mathrm{d}x$ and potential $P(t) = \frac{1}{2}\int_{\mathbb{R}} u_x(x,t)^2 \mathrm{d}x$ energies. Show that $K(t) = P(t)$ for all t sufficiently large.

Solution:

(a) By d'Alembert's formula

$$\begin{aligned} u(-x,t) &= \frac{1}{2}\left[\phi(-x+ct) + \phi(-x-ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\mathrm{d}s \\ &= \frac{1}{2}\left[-\phi(x-ct) - \phi(x+ct)\right] - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-s)\mathrm{d}s \\ &= -\left(\frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\mathrm{d}s\right) = -u(x,t) \end{aligned}$$

(b) $u \equiv 0$ is a solution since $u(x,0) \equiv 0 \equiv \phi(x)$; $u_t(x,0) \equiv 0 \equiv \psi(x)$; $u_{tt} = 0 = u_{xx}$. Define the energy $E(t) = \frac{1}{2}\int_{\mathbb{R}} u_t(x,t)^2 + c^2 u_x(x,t)^2 \mathrm{d}x = \bar{E}$, constant due to conservation. Thus, since $E(0) = \frac{1}{2}\int_{\mathbb{R}} \phi(x)^2 + c^2\psi(x)^2 \mathrm{d}x = 0 = \bar{E}$. Since $u_t^2, u_x^2 \geq 0$, this implies $u_t \equiv u_x \equiv 0$, and since $u(x,0) \equiv 0$, then $u(x,t) \equiv 0$.

(c) By d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[\phi(x+t) + \phi(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\mathrm{d}s$$

so

$$u_t(x,t) = \frac{1}{2}\left[\phi'(x+t) - \phi'(x-t)\right] + \frac{1}{2}\left[\psi(x+t) + \psi(x-t)\right]$$

and

$$u_x(x,t) = \frac{1}{2}\left[\phi'(x+t) + \phi'(x-t)\right] + \frac{1}{2}\left[\psi(x+t) - \psi(x-t)\right].$$

Now, since both ϕ and ψ have compact support, for sufficiently large t, all terms

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \phi'(x+t)\phi'(x-t)\mathrm{d}x \\
&\equiv \int_{\mathbb{R}} \phi'(x+t)\psi(x-t)\mathrm{d}x \\
&\equiv \int_{\mathbb{R}} \phi'(x-t)\psi(x+t)\mathrm{d}x \\
&\equiv \int_{\mathbb{R}} \psi(x-t)\psi(x+t)\mathrm{d}x.
\end{aligned}
$$

That is, if $\phi(x)$ is supported on $x \in [a,b]$ and $\psi(x)$ is supported on $x \in [m,n]$, then once $t > \max(n-m, b-a, n-a, b-m)/2 = t_0$, the length $2t$ will be wider than the region containing the support of both functions, so $\phi'(x)$ and $\psi(x)$ must be zero either at $x+t$ or $x-t$. Thus, we compute

$$
\begin{aligned}
K(t) = \frac{1}{8}\int_{\mathbb{R}} \Big[&\phi'(x+t)^2 + \phi'(x-t)^2 + 2\phi'(x+t)\psi(x+t) \\
&- 2\phi'(x-t)\psi(x-t) + \psi(x+t)^2 + \psi(x-t)^2\Big]\mathrm{d}x.
\end{aligned}
$$

For any finite $t > t_0$, we can change variables of all terms $z = x+t$ and $z = x-t$ and not change the integral, so

$$
K(t) = \frac{1}{4}\int_{\mathbb{R}} \left[\phi'(z)^2 + \psi(z)^2\right]\mathrm{d}z.
$$

Similarly,

$$
\begin{aligned}
P(t) = \frac{1}{8}\int_{\mathbb{R}} \Big[&\phi'(x+t)^2 + \phi'(x-t)^2 + 2\phi'(x+t)\psi(x+t) \\
&- 2\phi'(x-t)\psi(x-t) + \psi(x+t)^2 + \psi(x-t)^2\Big]\mathrm{d}x.
\end{aligned}
$$

and changing variables,

$$
P(t) = \frac{1}{4}\int_{\mathbb{R}} \left[\phi'(z)^2 + \psi(z)^2\right]\mathrm{d}z,
$$

so $K(t) = P(t)$ for $t > t_0$.

5. **(Method of characteristics)** Consider the quasilinear equation

$$(y+u)u_x + yu_y = x - y.$$

(a) Give an example of a connected curve $\Gamma \subset \mathbb{R}^2$ such that the Cauchy problem with prescribed data on that curve cannot be solved.

(b) Given the Cauchy data $u(x,1) = 1 + x$. What are the characteristic curves? Find the solution. For what values of $(x,y) \in \mathbb{R}^2$ does the solution exist?

Solution:

(a) We require the data $u(x_0(s), y_0(s)) = u_0(s)$, $s \in I$ to lie on a characteristic, i.e., for the vector field $\mathbf{v} = (y+u, y)$ to be tangent to the initial curve. For example, $\mathbf{v}$ is tangent to the curve $\Gamma = \{(x,0) \mid x \in \mathbb{R}\}$ for any choice of initial data. Note that the zero Jacobian condition

$$J = \begin{vmatrix} x_0'(s) & y_0'(s) \\ y_0(s) + u_0(s) & y_0(s) \end{vmatrix} = x_0' y_0 - y_0'(y_0 + u_0) = 0 \quad \text{for all } s \in I,$$

is necessary but not sufficient to prove that the problem cannot be solved.

(b) We parametrize the data as $x = s$, $y = 1$, $u = 1 + s$ for $s \in \mathbb{R}$. Then the characteristic equations for $x(\xi, s)$, $y(\xi, s)$, and $u(\xi, s)$ are

$$\begin{aligned} x_\xi &= y + u, \quad x(0,s) = s, \\ y_\xi &= y, \quad y(0,s) = 1, \\ u_\xi &= x - y, \quad u(0,s) = 1 + s, \quad s \in \mathbb{R}. \end{aligned}$$

Taking another derivative of the first equation, we obtain

$$\begin{aligned} x_{\xi\xi} &= y_\xi + u_\xi \\ &= y + x - y \\ &= x, \end{aligned}$$

which has solution $x = A(s)e^{\xi} + B(s)e^{-\xi}$. The initial data imply $x(0,s) = A(s) + B(s) = s$ and $x_\xi(0,s) = A(s) - B(s) = y(0,s) + u(0,s) = 2 + s$. Then $A(s) = 1 + s$ and $B(s) = -1$ so that

$$x(\xi, s) = (1+s)e^{\xi} - e^{-\xi}.$$

The equation for y is solved

$$y(\xi, s) = e^{\xi}.$$

Then the characteristic curves are

$$x = (1+s)y - \frac{1}{y}$$

for each $s \in \mathbb{R}$. We solve the equation for u

$$u_\xi = se^{\xi} - e^{-\xi} \quad \Rightarrow \quad u(\xi, s) = se^{\xi} + e^{-\xi}.$$

Undoing the characteristic transformation, we obtain the solution

$$u(x, y) = x - y + \frac{2}{y}, \quad x \in \mathbb{R}, \quad y \neq 0,$$

which exists for $y \neq 0$.