Preliminary Examination: Partial Differential Equations, 10 AM - 1 PM, Jan. 18, 2016, Discovery Learning Center (DLC) 1B70 (Bechtel Collaboratory).

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

Student ID:__________________________________

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points. A sheet of convenient formulae is provided.

1. **(Solution methods)** Let $\Omega = (0, \pi) \times (0, T)$, $T > 0$. Consider the initial, boundary value problem

$$\begin{aligned} u_t &= 2u_{xx}, \quad 0 < x < \pi, \quad t > 0 \\ u_x(\pi, t) = 2, \quad u(0, t) &= 0, \qquad t > 0, \\ u(x, 0) &= \phi(x), \quad 0 < x < \pi. \end{aligned}$$

   (a) Find a formal solution $u(x, t)$ that *solves* the above initial value problem.

   (b) Find sufficient conditions on ϕ such that the formal solution u is classical, i.e., it is in $\mathcal{C}_1^2(\bar{\Omega})$, functions that are twice continuously differentiable for $x \in [0, \pi]$ and continuously differentiable for $t \in [0, T]$. For full credit, you must provide a complete proof of your conclusion.

2. **(Heat equation)** Consider the following initial-boundary value problem for the heat equation

$$\begin{cases} u_t = u_{xx}, & x \in (0,1), \quad t > 0, \\ u(x,0) = x(1-x), & x \in (0,1), \\ u(0,t) = u(1,t) = 0, & t > 0. \end{cases}$$

Assume the existence of a classical solution $u(x,t)$.

(a) Prove the uniqueness of this solution.

(b) Show that $u(x,t) > 0$ on $x \in (0,1)$ and $t > 0$.

(c) For each $t > 0$, let $\mu(t) := \max_{x \in [0,1]} u(x,t)$. Show that $\mu(t)$ is a nonincreasing function of t.

3. **(Green's function)** Consider the boundary value problem

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^3, \\ u(\mathbf{x}) &= g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \end{aligned} \tag{1}$$

(a) Formulate a boundary value problem for Green's function $G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \phi^x(\mathbf{y})$ for $\mathbf{x} \in \Omega$ using the fundamental solution $\Phi(\mathbf{x}) = (4\pi|\mathbf{x}|)^{-1}$.

(b) Prove that Green's function, if it exists, is unique.

(c) Construct Green's function when

$$\Omega = B(0,1) \cap \left\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\right\},$$

where $B(0,1)$ is the unit sphere.

4. **(Wave equation)** Consider the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Assume the existence of a classical solution $u(x,t)$.

(a) If $\phi(x)$ and $\psi(x)$ are both odd functions of x, show that the solution $u(x,t)$ is odd in $x \in \mathbb{R}$ for $t > 0$.

(b) Find a solution to Eq. (2) assuming $\phi(x) \equiv 0$ and $\psi(x) \equiv 0$, and prove that it is unique.

(c) Assume $\phi(x)$ and $\psi(x)$ have compact support, fix $c = 1$, and define the kinetic $K(t) = \frac{1}{2} \int_{\mathbb{R}} u_t(x,t)^2 \mathrm{d}x$ and potential $P(t) = \frac{1}{2} \int_{\mathbb{R}} u_x(x,t)^2 \mathrm{d}x$ energies. Show that $K(t) = P(t)$ for all t sufficiently large.

5. **(Method of characteristics)** Consider the quasilinear equation

$$(y + u)u_x + yu_y = x - y.$$

(a) Give an example of a connected curve $\Gamma \subset \mathbb{R}^2$ such that the Cauchy problem with prescribed data on that curve cannot be solved.

(b) Given the Cauchy data $u(x, 1) = 1 + x$. What are the characteristic curves? Find the solution. For what values of $(x, y) \in \mathbb{R}^2$ does the solution exist?