PDE Preliminary Examination: 1/14/2015

Name:____________________________________

There are 5 problems, each worth 25 points. You are required to do 4 of them. Indicate in the table which 4 you choose–Note: Only 4 problems will be graded. A sheet of convenient formulae is provided.

#	Choice (X)	score
1		
2		
3		
4		
5		
Total		

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1. **Heat Equation**

Let $Q = (0, \pi) \times (0, T)$ and $\overline{Q}$ the closure of this domain. Suppose that $u(x,t) \in C^2(Q) \times C^0(\overline{Q})$ is a solution to:

$$\begin{array}{lll} u_t(x,t) = u_{xx}(x,t) + F(x,t), & (x,t) \in Q, & \\ u(0,t) = g(t), \quad u(\pi,t) = 0, & t > 0, & \\ u(x,0) = f(x), & 0 \le x \le \pi. & \end{array} \tag{1}$$

(a) Let $M = \max\{0, g(t), f(x) | (x,t) \in \overline{Q}\}$, $N = \max\{0,\ F(x,t) | (x,t) \in \overline{Q}\}$, show that $u(x,t) \le M + tN$. (State clearly the theorems that you are using).

(b) Let $g \equiv 0$ and $F \equiv 0$. It is known that when $f'(x)$ and $f(x)$ are continuous on $[0, \pi]$ with $f(0) = f(\pi) = 0$, the above equation has a classical solution (a solution $u(x,t) \in C^2(Q) \times C^0(\overline{Q})$). Show the existence and uniqueness of a classical solution when f is continuous and $f(0) = f(\pi) = 0$.

Solution a) Notice that $N \ge 0$. Let $w = u - (M + Nt)$. Then $w_t - w_{xx} = F(x,t) - N \le 0$, and $w(0,t) = u(0,t) - M - Nt \le u(0,t) - M \le 0$, $w(L,t) \le 0$ likewise, also $w(x,0) \le 0$.

By the maximum principle, $w(x,t) \le 0$, or equivalently, $u(x,t) \le M + tN$.

b) It is clear from the formal solution $u(x,t)$ (via separation of variables) that we get a smooth solution for $t > 0$. The key is to prove that this solution can be continuously extend to $t = 0$.

For this purpose, we take a sequence of functions $f_n(x) \in C^1[0, \pi]$ with $f_n(0) = f_n(L) = 0$ that converge uniformly to $f(x)$. Then the corresponding formal solution $u_n(x,t)$ is in $C^2(Q) \times C^0(\overline{Q})$. By (a), we can get:

$$||u_n - u_m||_{C^0(\overline{Q})} \le ||f_n - f_m||_{C^0[0,\pi]} \triangleq \epsilon_{n,m} \to 0$$

as $n, m \to \infty$. Thus, $\exists \overline{u} \in C^0(\overline{Q})$ such that u_n converges uniformly to $\overline{u}$ on $\overline{Q}$.

Clearly, by the bounded convergence theorem, $\overline{u} = u$ for $t > 0$. In other words, the solution $u(x,t)$, originally defined for $t > 0$ can be continuously extended to $t = 0$ with $u(x,0) \triangleq \lim_{t \to 0+} u(x,t) = \lim_{t \to 0+} \overline{u}(x,t) = \overline{u}(x,0) = f(x)$.

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2. **Fourier Series**

(a) Prove the Weierstrass approximation theorem: let $f(x)$ be a 2π-periodic, continuous function, then $\forall \epsilon > 0$, there exists a trigonometric polynomial $T(x)$, such that $|f(x) - T(x)| \leq \epsilon$, $\forall x \in \mathbb{R}$. (hint: construct a suitable reproducing kernel/approximation of identity).

(b) Prove Parseval's identity: if $f(x)$ is a 2π-periodic, continuous function and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

is its Fourier Series, then:

$$\int_{-\pi}^{\pi} f^2(x)dx = \pi\frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Solution

Solution for part (a). Step 1 $\varphi_n(u) = c_n \cos^{2n} \frac{u}{2}$ where $c_n = \left(\int_{-\pi}^{\pi} \cos^{2n} \frac{u}{2} du\right)^{-1}$ is an approximate identity on $[-\pi, \pi]$, and it is a trigonometric polynomial of degree n in u.

Proof. We must show that φ_n is trigonometric polynomial and an approximate identity.

(a) Writing $\varphi_n(u) = \frac{c_n}{2^{2n}}(e^{\frac{iu}{2}} + e^{\frac{-iu}{2}})^{2n}$ with $c_n^{-1} = \int_{-\pi}^{\pi} \cos^{2n} \frac{u}{2} du$

(b) To show that φ_n is an approximate identity, we must verify the three properties of an approximate identity:

i.) $\varphi_n(u) \geq 0$ since $\cos \frac{u}{2} \geq 0$ for $|u| \leq \pi$.

ii.)

$$\int_{-\pi}^{\pi} \varphi_n(u)du = \frac{\int_{-\pi}^{\pi} \cos^{2n} \frac{u}{2} du}{\int_{-\pi}^{\pi} \cos^{2n} \frac{u}{2} du} = 1$$

iii.) $\varphi_n(u)$ is 'smooth' (C^∞ in fact), and $\forall \delta > 0$

$$0 \leq \lim_{n\to\infty} \int_{|u|\geq\delta} \varphi_n(u)du \leq \lim_{n\to\infty} \frac{\int_{|u|\geq\delta} \cos^{2n} \frac{u}{2} du}{\int_{|u|\leq\frac{\delta}{2}} \cos^{2n} \frac{u}{2} du} \leq \lim_{n\to\infty} \frac{2\pi}{\delta}\left(\frac{\cos\frac{\delta}{2}}{\cos\frac{\delta}{4}}\right)^{2n} = 0$$

□

Thus $\lim_{n\to\infty} \int_{|u|\geq\delta} \varphi_n(u) = 0$ by the squeeze theorem.

Step 2: $T_n(x) = \int_{-\pi}^{\pi} f(x+u)\varphi_n(u)du$ is a trigonometric polynomial of degree at most n, such that

$$a_n = \max |T_n(x) - f(x)| \to 0$$

(i)

$$\begin{aligned} T_n(x) &= \int_{-\pi}^{\pi} f(x+u)\varphi_n(u)du \\ &= \int_{-\pi+x}^{\pi+x} f(\omega)\varphi_n(\omega - x)d\omega \\ &= \int_{-\pi}^{\pi} f(\omega)\varphi_n(\omega - x)d\omega \end{aligned}$$

since both $f(\omega)$ and $\varphi_n(\omega)$ are 2π-periodic. $\varphi_n(\omega - x)$ is a trigonometric polynomial of degree n in $\omega - x$ and by the difference angle theorem of cosine and sine, we have $\varphi_n(\omega - x)$ is a trigonometric polynomial of degree $\leq n$ in x with coefficients as functions in ω. Thus $T_n(x) = \int_{-\pi}^{\pi} f(\omega)\varphi_n(\omega - x)dx$ is trigonometric polynomial of degree $\leq n$ in x.

(ii) By the theorem of approximate identity, $T_n(x)$ converges to $f(x)$ uniformly, or, $\forall \epsilon > 0\ \exists N$, such that

$$|f(x) - T_n(x)| \leq \epsilon \quad \forall x \in \mathbb{R}$$

□

Solution for part (b). Let

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

then by the definition of a_0, a_k, and b_k, we know S_n is the projection of $f(x)$ into

$$H_n = \text{span}\{1, \cos kx, \sin kx\}_{k=1}^{n}$$

in the Hilbert space $L^2(-\pi, \pi)$. Thus for any trigonometric polynomial $T(x)$ of degree $\leq n$ (or equivalently, $T(x) \in H_n$), we must have

$$||f(x) - S_n(x)||_{L^2} \leq ||f(x) - T(x)||$$

Step 1 $\lim_{n\to\infty} ||f(x) - S_n(x)||_{L^2} = 0$ via Weierstrass. For all $\epsilon > 0$, $\exists T(x)$, a trigonometric polynomial, such that

$$|f(x) - T(x)| \leq \frac{\epsilon}{\sqrt{2\pi}}$$

which implies

$$\begin{aligned} \int_{-\pi}^{\pi} |f(x) - T(x)|^2 dx &\leq \frac{\epsilon^2}{2\pi} \int_{-\pi}^{\pi} dx \\ &= \epsilon^2 \end{aligned}$$

and thus

$$||f(x) - T(x)||_{L^2} \le \epsilon$$

Let $N = \deg(T(x))$, then for any $n \ge N$, $T(x) \in H_n$ implies

$$\begin{aligned} ||f(x) - S_n(x)||_{L^2} &\le ||f(x) - T(x)|| \\ &\le \epsilon \end{aligned}$$

or

$$\lim_{n \to \infty} ||f(x) - S_n(x)||_{L^2} = 0$$

Step 2

$$\int_{-\pi}^{\pi} f^2(x)dx = \frac{a_0^2}{2}\pi + \pi \sum_{n=1}^{\infty}(a^2 + b^2)$$

Using the fact that

$$\int_{-\pi}^{\pi} \cos^2 nx dx = \int_{-\pi}^{\pi} \sin^2 nx dx = \pi$$

and the orthogonality, we see

$$||S_n||_{L^2}^2 = \int_{-\pi}^{\pi} S_n^2(x)dx = \frac{a_0^2}{2}\pi + \pi \sum_{n=1}^{N}(a^2 + b^2)$$

Thus from step 1,

$$\begin{aligned} \int_{-\pi}^{\pi} f^2(x)dx = ||f||_{L^2}^2 &= \lim_{n \to \infty} ||S_n||_{L^2}^2 \\ &= \frac{a_0^2}{2}\pi + \lim_{N \to \infty} \pi \sum_{n=1}^{N}(a^2 + b^2) \\ &= \frac{a_0^2}{2}\pi + \pi \sum_{n=1}^{\infty}(a^2 + b^2) \end{aligned}$$

□

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3. **Method of Characteristics.** The nonlinear PDE

$$v_{tt}v_x^2 - 2v_{xt}v_tv_x + v_t^2v_{xx} = 0 \tag{2}$$

is a special case of the so-called *Monge-Ampère* equation. In this problem, you will reduce this system to an equivalent first order equation and then solve it.

(a) Show that (2) is equivalent to:

$$\frac{v_{tt}}{v_x} - \frac{v_t v_{xt}}{v_x^2} = \frac{v_t}{v_x}\left\{\frac{v_{xt}}{v_x} - \frac{v_t v_{xx}}{v_x^2}\right\} \tag{3}$$

Then show that (3) can be written as an equivalent first order PDE for the new function $u = v_t/v_x$. [Hint: we ordered the terms in (3) for a reason!]

(b) For the given initial conditions

$$\begin{aligned} v(x,0) &= 1 + 2e^{3x} \\ v_t(x,0) &= 4e^{3x} \end{aligned}$$

on $-\infty < x < \infty$, find $u(x,t)$ for $t > 0$ and then find $v(x,t)$ for $t > 0$.

Solution (a) It is straightforward to divide by v_x^3 and rearrange to get the result (3). Substitution for $u = v_t/v_x$ then gives

$$u_t - uu_x = 0$$

(b) For the given initial/boundary conditions on v, we have

$$u_0(x) = u(x,0) = \frac{v_t(x,0)}{v_x(x,0)} = \frac{4e^3x}{2 \cdot 3e^{3x}} = \frac{2}{3}$$

To solve by the MoC, we write $U(t;x_0) = u(X(t;x_0),t)$, and then solve

$$\begin{aligned} \frac{dX}{dt} &= -U \\ \frac{dU}{dt} &= 0 \end{aligned}$$

subject to $U(0;x_0) = u_0(x_0)$ and $X(0;x_0) = x_0$. This gives $U(t;x_0) = u_0(x_0)$, and $X(t;x_0) = -u_0(x_0)t + x_0$, which gives a formal solution to the PDE

$$u(x,t) = u_0\left(x + u_0(x_0(t;x))dt\right),$$

though we would have to invert $x(t;x_0)$ to find $x_0(t;x)$. However, since $u_0 = \frac{2}{3}$, this simply gives $u(x,t) = \frac{2}{3}$ too.

To solve for v, we must solve the PDE

$$v_t - uv_x = v_t - \tfrac{2}{3}v_x = 0$$

This also easy, since it is now linear with constant coefficients, $v(x,t) = v_0(x + \frac{2}{3}t)$, so since $v_0(x) = v(x,0)$ we have

$$v(x,t) = 1 + 2e^{3x+2t}$$

Note that $v_t(x,0) = 4e^{3x}$, as required. We check by substitution back into (2); this gives an identity.

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4. **Wave equation**

(a) Let u be a classical solution of $u_{tt} = c^2 u_{xx}$ $(c > 0)$ on $\mathbb{R} \times (0, \infty)$ and define

$$E_{x,t}(s) = \frac{1}{2} \int_{x-c(t-s)}^{x+c(t-s)} [\ u_s^2(y,s) + c^2 u_y^2(y,s)]dy$$

for $x \in \mathbb{R}$ and $t \geq s > 0$. Show that $\frac{d}{ds} E_{x,t}(s) \leq 0$ for $s \in (0,t)$.

(b) For the classical solution in (a), let $E(s) = \frac{1}{2} \int_{-\infty}^{\infty} [u_s^2(y,s) + c^2 u_y^2(y,s)]dy$. Show that $E(s)$ is monotone non-increasing, and in particular, if $E(s_0)$ is finite then show that $E(s)$ is finite for all $s > s_0$.

(c) Apply the 'energy inequality' from (a) to show that there is at most one classical solution to the initial value problem:

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t) + F(x,t), \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned}$$

with $c > 0$, such that $u(x,t) \in C^1(\mathbb{R} \times [0,\infty)) \cap C^2(\mathbb{R} \times (0,\infty))$.

Solution for part (a).

$$\begin{aligned} \frac{d}{ds} E_{x,t}(s) &= -c[\tfrac{1}{2} u_s^2(x + c(t-s)), s) + \tfrac{1}{2} c^2 u_y^2(x + c(t-s), s)] \\ &\quad -c[\tfrac{1}{2} u_s^2(x - c(t-s)), s) + \tfrac{1}{2} c^2 u_y^2(x - c(t-s), s)] \\ &\quad + \textstyle\int_{x-c(t-s)}^{x+c(t-s)} [u_s(y,s) u_{ss}(y,s) + c^2 u_y(y,s) u_{ys}(y,s)]dy \\ &= -c[\tfrac{1}{2} u_s^2(x + c(t-s)), s) + \tfrac{1}{2} c^2 u_y^2(x + c(t-s), s)] \\ &\quad -c[\tfrac{1}{2} u_s^2(x - c(t-s)), s) + \tfrac{1}{2} c^2 u_y^2(x - c(t-s), s)] \\ &\quad +c^2 u_y(x + c(t-s), s) u_s(x + c(t-s), s) \\ &\quad -c^2 u_y(x - c(t-s), s) u_s(x - c(t-s), s) \\ &\quad + \textstyle\int_{x-c(t-s)}^{x+c(t-s)} [u_s(y,s) u_{ss}(y,s) - c^2 u_s(y,s) u_{yy}(y,s)]dy \\ &= -\tfrac{c}{2}[u_s(x + c(t-s)), s) - c u_y(x + c(t-s), s)]^2 \\ &\quad -\tfrac{c}{2}[u_s(x - c(t-s)), s) + c u_y(x - c(t-s), s)]^2 \\ &\leq 0 \end{aligned}$$

□ □

Solution for part (b). First, since the integrand does not change sign, for fixed s and any given x, we see that $\lim_{t \to +\infty} E_{x,t}(s) = \lim_{t \to +\infty} \frac{1}{2} \int_{x-c(t-s)}^{x+c(t-s)} [\ u_s^2(y,s) + c^2 u_y^2(y,s)]dy = \frac{1}{2} \int_{-\infty}^{\infty} [u_s^2(y,s) + c^2 u_y^2(y,s)]dy \triangleq E(s)$. □

Thus for $0 \le s_1 < s_2$ we have $E(s_2) = \lim_{t\to+\infty} E_{0,t}(s_2) = \lim_{t\to+\infty} \frac{1}{2}\int_{-c(t-s_2)}^{c(t-s_2)}[\ u_s^2(y,s_2) + c^2u_y^2(y,s_2)]dy \le \lim_{t\to+\infty} \frac{1}{2}\int_{-c(t-s_1)}^{c(t-s_1)}[\ u_s^2(y,s_1) + c^2u_y^2(y,s_1)]dy = E(s_1)$. Here we have pick $x = 0$.

In short, $E(s_2) \le E(s_1)$ for $s_1 < s_2$ or $E(s)$ is monotone non-increasing and consequently if $E(0)$ is finite then $E(s)$ is also finite for all $s \ge 0$.

Solution for part (c). Suppose u_1, u_2 are two solutions. Let $u = u_1 - u_2$. Note that this solves the homogeneous problem, with $F = f = g = 0$. By part (a), $E_{x,t}(s) \le E_{x,t}(0) = 0$. So, since both terms in the integrand are non-negative,

$$\int_{x-ct}^{x+ct} u_s^2(y,s)dy = 0.$$

Since $u_s(y,s)$ is continuous, we have $u_s(y,s) = 0$ for $y \in (x-ct, x+ct)$, $0 < s \le t$. Now, let $t \to \infty$, and we have $u_s(y,s) \equiv 0$ for $y \in (-\infty,\infty)$ and $s > 0$. Since $u(y,0) = 0$, we get $u(y,s) \equiv 0$. □

5. **Poisson's Equation**

Let $B = \{(r,\theta) \,|\, 0 \le r < a,\ 0 \le \theta < 2\pi\} \in \mathbb{R}^2$ for $a > 0$, be the open disk of radius a centered at the origin, with polar coordinates (r,θ). Consider the problem

$$\begin{aligned} \Delta u &= F(r,\theta), \quad (r,\theta) \in B, \\ u(a,\theta) &= f(\theta). \end{aligned} \tag{4}$$

(a) Find a formal solution $u(r,\theta)$ that 'solves' (4) for $F \equiv 0$.

(b) Find conditions on f that assure the formal solution u obtained in part (a) is in $C^0(\overline{B})$. Give a proof of your conclusion.

(c) State and prove a version of the maximum principle (stability estimate) for (4). Apply it to prove that (4) admits at most one classical solution $u \in C^0(\overline{B}) \cap C^2(B)$ for given functions F and f.

Solution for part (a). :

The formal solution is given by (you can derive this either by separation of variables or using the analytic functions z^n):

$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)(\frac{r}{a})^n$, with

$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos(n\theta)f(\theta)d\theta$ and $b_n = \frac{1}{\pi}\int_0^{2\pi} \sin(n\theta)f(\theta)d\theta$ for $n = 0, 1, 2, 3, \cdots$.

□

Solution for part (b). The formal solution is in $C^0(\overline{B})$ if f is in $C^1[0,2\pi]$ with $f(0) = f(2\pi)$.

In fact, let $f'(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n \cos n\theta + \beta_n \sin n\theta)$.

Then by Parseval's identity, we have $\sum_{n=1}^{\infty}(\alpha_n^2 + \beta_n^2) < \infty$.

Applying the integration by parts, and using the fact that $f(0) = f(2\pi)$, we derive: $a_n = \frac{-\beta_n}{n}$ and $b_n = \frac{\alpha_n}{n}$. Thus, each term in our formal solution satisfies:

$$|(a_n \cos n\theta + b_n \sin n\theta)(\frac{r}{a})^n| \leq \frac{|\beta_n| + |\alpha_n|}{n}$$

To show that the formal series solution give a continuous function for $r \leq a$, we only need to check that the dominating convergence theorem can be applied here for uniform convergence:

We only need: $\sum_{n=1}^{\infty} \frac{|\beta_n|+|\alpha_n|}{n} < \infty$. This can be obtain via Schwartz inequality: $\sum_{n=1}^{\infty} \frac{|\beta_n|+|\alpha_n|}{n} \leq (\sum_{n=1}^{\infty} |\beta_n|^2)^{\frac{1}{2}}(\sum_{n=1}^{\infty} \frac{1}{n^2})^{\frac{1}{2}} + (\sum_{n=1}^{\infty} |\alpha_n|^2)^{\frac{1}{2}}(\sum_{n=1}^{\infty} \frac{1}{n^2})^{\frac{1}{2}} < \infty$.

The proof is completed.

□

Solution for part (c). Statement of a MP:

Assume that $u \in C^2(B) \bigcap C(\bar{B})$ and $\triangle u = F$ in B, then $|u(x)| \leq N + \frac{a^2}{4}M$ where $M = \max_{\bar{B}} |F|$ and $N = \max_{\partial B} |u| = \max |f|$. In particular, when $F \equiv 0$ in B and $u \equiv 0$ on ∂B, then $u \equiv 0$ on B.

This is equivalent to the uniqueness of the Dirichlet problem:

$$u \in C^2(B) \bigcap C(\bar{B})$$
$$\Delta u = F \text{ in } B$$
$$u = f \text{ on } \partial B.$$

Proof of the MP:

Define $w(x) = u(x) + \frac{(a^2-r^2)}{4}M + N$. We see that $\triangle w = F - M \leq 0$ so $\min_{\bar{B}} w = \min_{\partial B} w \geq 0$ or $w \geq 0$ in B. Equivalently, we get $u(x) \geq -(N + \frac{a^2}{4}M) + \frac{r^2}{4}M \geq -(N + \frac{a^2}{4}M)$.

Similarly, let $w = u(x) - (\frac{(a^2-r^2)}{4}M + N)$. We see that $\triangle w = h + H \geq 0$ so $\max_{\bar{B}} w = \max_{\partial B} w \leq 0$ or $w \leq 0$ in B. Equivalently, we get $u(x) \leq (N + \frac{a^2}{4}M) - \frac{r^2}{4}M \geq (N + \frac{a^2}{4}M)$.

Combining the above two estimates, we get $|u(x)| \leq N + \frac{a^2}{4}M$.

Uniqueness:

If u_1 and u_2 are two solutions in $C^2(B) \bigcap C(\bar{B})$, we can apply the above to $w = u_1 - u_2$ with corresponding $M = N = 0$ and thus $w \equiv 0$. Consequently, $u_1 \equiv u_2$. □