#	Choice (X)	score
1		
2		
3		
4		
5		
Total		

PDE Preliminary Examination: 1/14/2015

Name:______________________________________

There are 5 problems, each worth 25 points. You are required to do 4 of them. Indicate in the table which 4 you choose–Note: Only 4 problems will be graded. A sheet of convenient formulae is provided.

---

1. **Heat Equation**

Let $Q = (0,\pi) \times (0,T)$ and $\overline{Q}$ the closure of this domain. Suppose that $u(x,t) \in C^2(Q) \times C^0(\overline{Q})$ is a solution to:

$$\begin{array}{lll} u_t(x,t) = u_{xx}(x,t) + F(x,t), & & (x,t) \in Q, \\ u(0,t) = g(t), & u(\pi,t) = 0, & t > 0, \\ u(x,0) = f(x), & & 0 \le x \le \pi. \end{array} \tag{1}$$

(a) Let $M = \max\{0, g(t), f(x) | (x,t) \in \overline{Q}\}$, $N = \max\{0,\ F(x,t) | (x,t) \in \overline{Q}\}$, show that $u(x,t) \le M + tN$. (State clearly the theorems that you are using).

(b) Let $g \equiv 0$ and $F \equiv 0$. It is known that when $f'(x)$ and $f(x)$ are continuous on $[0,\pi]$ with $f(0) = f(\pi) = 0$, the above equation has a classical solution (a solution $u(x,t) \in C^2(Q) \times C^0(\overline{Q})$). Show the existence and uniqueness of a classical solution when f is continuous and $f(0) = f(\pi) = 0$.

---

2. **Fourier Series**

(a) Prove the Weierstrass approximation theorem: let $f(x)$ be a 2π-periodic, continuous function, then $\forall \epsilon > 0$, there exists a trigonometric polynomial $T(x)$, such that $|f(x) - T(x)| \le \epsilon$, $\forall x \in \mathbb{R}$. (hint: construct a suitable reproducing kernel/approximation of identity).

(b) Prove Parseval's identity: if $f(x)$ is a 2π-periodic, continuous function and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

is its Fourier Series, then:

$$\int_{-\pi}^{\pi} f^2(x)dx = \pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

…Turn over for problem 3…

3. **Method of Characteristics.** The nonlinear PDE

$$v_{tt}v_x^2 - 2v_{xt}v_tv_x + v_t^2v_{xx} = 0 \tag{2}$$

is a special case of the so-called *Monge-Ampère* equation. In this problem, you will reduce this system to an equivalent first order equation and then solve it.

(a) Show that (2) is equivalent to:

$$\frac{v_{tt}}{v_x} - \frac{v_tv_{xt}}{v_x^2} = \frac{v_t}{v_x}\left\{\frac{v_{xt}}{v_x} - \frac{v_tv_{xx}}{v_x^2}\right\} \tag{3}$$

Then show that (3) can be written as an equivalent first order PDE for the new function $u = v_t/v_x$. [Hint: we ordered the terms in (3) for a reason!]

(b) For the given initial conditions

$$\begin{aligned} v(x,0) &= 1 + 2e^{3x} \\ v_t(x,0) &= 4e^{3x} \end{aligned}$$

on $-\infty < x < \infty$, find $u(x,t)$ for $t > 0$ and then find $v(x,t)$ for $t > 0$.

---

4. **Wave equation**

(a) Let u be a classical solution of $u_{tt} = c^2u_{xx}$ ($c > 0$) on $\mathbb{R} \times (0,\infty)$ and define

$$E_{x,t}(s) = \frac{1}{2}\int_{x-c(t-s)}^{x+c(t-s)} [\, u_s^2(y,s) + c^2u_y^2(y,s)]dy$$

for $x \in \mathbb{R}$ and $t \geq s > 0$. Show that $\frac{d}{ds}E_{x,t}(s) \leq 0$ for $s \in (0,t)$.

(b) For the classical solution in (a), let $E(s) = \frac{1}{2}\int_{-\infty}^{\infty}[u_s^2(y,s) + c^2u_y^2(y,s)]dy$. Show that $E(s)$ is monotone non-increasing, and in particular, if $E(s_0)$ is finite then show that $E(s)$ is finite for all $s > s_0$.

(c) Apply the 'energy inequality' from (a) to show that there is at most one classical solution to the initial value problem:

$$\begin{aligned} u_{tt}(x,t) &= c^2u_{xx}(x,t) + F(x,t), \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned}$$

with $c > 0$, such that $u(x,t) \in C^1(\mathbb{R} \times [0,\infty)) \cap C^2(\mathbb{R} \times (0,\infty))$.

5. **Poisson's Equation**

Let $B = \{(r,\theta)\,|\,0 \le r < a,\ 0 \le \theta < 2\pi\} \in \mathbb{R}^2$ for $a > 0$, be the open disk of radius a centered at the origin, with polar coordinates (r,θ). Consider the problem

$$\begin{aligned} \Delta u &= F(r,\theta), \quad (r,\theta) \in B, \\ u(a,\theta) &= f(\theta). \end{aligned} \tag{4}$$

(a) Find a formal solution $u(r,\theta)$ that 'solves' (4) for $F \equiv 0$.

(b) Find conditions on f that assure the formal solution u obtained in part (a) is in $C^0(\overline{B})$. Give a proof of your conclusion.

(c) State and prove a version of the maximum principle (stability estimate) for (4). Apply it to prove that (4) admits at most one classical solution $u \in C^0(\overline{B}) \cap C^2(B)$ for given functions F and f.