PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
January 2014

You have three hours to complete this exam. Each problem is worth 25 points. Work only four of the five problems problems. Please mark which four that you choose—only four will be graded. Please start each problem on a new page. A sheet of convenient formulae is attached.

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1. Method of Characteristics.

   Let $u(x,t)$ solve the following "forced Burgers" equation

$$\partial_t u + u\ \partial_x u = f(u)$$

   with initial data

$$u(x,0) = \begin{cases} 1 & \text{for} \quad x \le 0 \\ 1-x & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x \ge 1 \end{cases}$$

   Consider the following two cases

$$f(u) = \begin{cases} +u & \text{Case } \alpha \\ -u & \text{Case } \beta \end{cases}$$

   a) Solve the above for cases α and β.
   b) Show that a shock, i.e a discontinuity, develops in finite time $t = t_* < \infty$ for case α only.

2. Fourier Series and Convergence.

   For parts (a) and (b), consider a piecewise smooth function $f(x)$ on $-\pi < x < \pi$. Let a_n and b_n be the Fourier coefficients of f.

   (a) Prove that a_n is $\mathcal{O}(1/n)$.
   (b) For $\lim_{x \searrow -L} f(x) = \lim_{x \nearrow L} f(x)$, prove that $|a_n|$ is actually $\ll 1/n$, i.e., $a_n = o(1/n)$. HINT: consider the relationship between the Fourier coefficients of $f(x)$ and the Fourier coefficients of $f'(x)$.
   (c) The Riemann-Lebesgue Lemma states that if g is continuous on $[a,b]$, except possibly at a finite number of points, then

$$\lim_{\lambda\to\infty} \int_a^b g(x)\sin(\lambda x)\mathrm{d}x = \lim_{\lambda\to\infty} \int_a^b g(x)\cos(\lambda x)\mathrm{d}x = 0\,.$$

   Prove this lemma.

3. Wave Equation.

   Using D'Alembert's approach, find all solutions $u(t,x)$ in the first quadrant $t,x>0$ of the initial value problem

$$\begin{aligned} u_{tt}-c^2u_{xx}=0 \quad &, \quad x>0,\, t>0 \\ u(0,x)=f(x) \quad &, \quad u_t(0,x)=g(x) \\ u_t(t,0)=au_x(t,0) \quad &, \end{aligned}$$

   where $f(x)$ and $g(x)$ are $\mathcal{C}^2$ functions that are zero near $x=0$. Be sure to specify any situations where a solution does not exist.

   HINT: Note that

$$\begin{aligned} u(x,t) &= F(x+ct)+G(x-ct) \\ F(x) &= \frac{1}{2}f(x)+\frac{1}{2c}\int_0^x g(s)ds \\ G(x) &= \frac{1}{2}f(x)-\frac{1}{2c}\int_0^x g(s)ds \end{aligned}$$

4. For the Laplace Equation:

   (a) Find the Green's function for the Dirichlet problem on the ball of radius 1 in $\mathbb{R}^2$. HINT: The fundamental solution for the Laplacian on $\mathbb{R}^2$ is $\Phi(x,x')=-\ln|x-x'|/(2\pi)$.

   (b) Write down and justify the formula for smooth solutions of

$$\begin{aligned} \Delta u=0 \quad &; \quad x\in\mathbb{R}^2,\, |x|<1 \\ u(x)=g(x) \quad &; \quad |x|=1 \end{aligned}$$

   where g is a smooth function on the unit circle.

   (c) Use the maximum principle to prove the uniqueness of the solution.

5. Heat Equation

(a) Given the equation

$$u_t - u_{xx} - u = 0, \quad |x| < \infty, \ t > 0$$

with $u(0, x) = \delta(x - x_0)$ and u and its derivatives vanish as $x \to \infty$. Find the solution u, *evaluate all integrals* and discuss important aspects of the solution.

(b) Given the equation

$$u_t - \nabla^2 u - u = 0, \quad |x| < \infty, \quad t > 0$$

where $x = (x_1, ...x_N)$, $\nabla^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial {x_i}^2}$ and $|x_i| < \infty, \ u(t = 0, x) = f(x)$ and u and its derivatives vanish at ∞.

Show the solution, assuming it exists, is unique.

(c) Given the equation

$$u_t - u_{xx} = 0, \quad |x| < \infty, \ t > 0$$

i. Find the most general “self-similar" solution of the form:

$$u = \frac{1}{\sqrt{2t}} \ F(\xi), \ \xi = \frac{x}{\sqrt{2t}}$$

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ii. Find the solution F that vanishes as $|x| \to \infty$

iii. Explain whether this solution is related to any special solution of the heat equation.