Preliminary Examination: Partial Differential Equations,
10:00 AM - 1:00 PM, Jan. 14, 2013,
Rooms KOBL 350 and KOBL 355.

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

Name:____________________________________

There are 5 problems. **Do problems 1, 2 and 3, and choose one between problems 4 and 5**. Each problem is worth 25 points. A sheet of convenient formulae is provided.

1. (a) State and prove the Riemann-Lebesgue Lemma.

   (b) Assume $f(x) : [-\pi, \pi] \to \mathbb{R}$ has continuous derivatives up to order k which satisfy

$$f^{(j)}(-\pi) = f^{(j)}(\pi), \qquad j = 0, 1, 2, \ldots, k-1, \tag{1}$$

   and let the Fourier series of $f(x)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]. \tag{2}$$

   Show that the Fourier coefficients satisfy

$$|a_n| \le \frac{\epsilon_n}{n^k}, \qquad |b_n| \le \frac{\epsilon_n}{n^k}, \qquad n = 1, 2, \ldots \tag{3}$$

   where $\epsilon_n > 0$ is such that the series $\sum_{n=1}^{\infty} \epsilon_n^2$ converges.

2. Let Ω be a bounded open subset of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$ having unit outward normal $\mathbf{n}$. Show that if $a > 0$, then the boundary value problem

$$\begin{aligned} \Delta u &= f(x), \quad x \in \Omega, \\ au + \frac{\partial u}{\partial n} &= g(x), \quad x \in \partial\Omega, \end{aligned}$$

   has at most one solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^2(\bar{\Omega})$.

TURN OVER

3. Let a curve be defined by $\gamma(\hat{x}) = \{(x,t) \in \mathbb{R}^2 | t = -x \text{ and } x \leq \hat{x}\}$ and consider the partial differential equation for $u(x,t)$ with initial conditions on this curve:

$$\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = u, \qquad t \geq -x, \tag{4}$$

$$u(s,-s) = \sin(s), \qquad s \leq \hat{x}. \tag{5}$$

(a) Find the largest x_{max} for which we can guarantee that the differential equation above has a unique solution defined in a neighborhood of $\gamma(\hat{x})$ for all $\hat{x} < x_{max}$. Justify your answer.

(b) Find the solution $u(x,t)$ in such a neighborhood.

(c) Find the values of (x,t) for which the solution $u(x,t)$ above is defined.

4. **(Do either problem 4 or problem 5.)** Let $\Omega = (0,\pi) \times (0,T)$, $T > 0$ and assume that $u(x,t) \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{C}^2(\Omega)$ and satisfies

$$u_t = 2u_{xx}, \quad 0 < x < \pi, \quad t > 0$$

$$u_x(0,t) = 0, \quad u(\pi,t) = 2, \qquad t > 0,$$

$$u(x,0) = \phi(x), \quad 0 < x < \pi.$$

(a) Find a formal solution $u(x,t)$ that *solves* the above initial value problem.

(b) Find nontrivial conditions on ϕ such that the formal solution u is in $\mathcal{C}^2(\bar{\Omega})$ and thus assure the existence of a *classical solution* under these conditions. For full credit, you must provide a proof of your conclusion.

5. **(Do either problem 4 or problem 5.)** Consider the bounded solution $u = u(r,\theta)$ to Laplace's equation in a wedge with radius a and angle θ_0

$$\Delta u = 0, \quad 0 < r < a, \quad 0 < \theta < \theta_0$$

and with boundary conditions

$$u(r,0) = \frac{\partial}{\partial \theta} u(r,\theta)\Big|_{\theta=\theta_0} = 0, \quad 0 < r < a$$

$$u(a,\theta) = g(\theta), \quad 0 < \theta < \theta_0\,,$$

for g a continuous and bounded function and $\theta_0 < \pi/4$.

(a) Find a formal solution $u(r,t)$ that *solves* the above boundary value problem.

(b) Find nontrivial conditions on g such that the formal solution u is in $\mathcal{C}^2(\bar{\Omega})$ and thus assure the existence of a *classical solution* under these conditions. For full credit, you must provide a proof of your conclusion.