PDE Preliminary Examination: Spring 2012

Name:________________________________

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are 5 problems. Each problem is worth 25 points. You are required to do 4 of them. Please indicate which 4 you choose–Note: Only 4 problems will be graded. A sheet of convenient formulae is provided.

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1. Heat Equation

Assume that $u(x,t) \in C(\overline{Q}) \bigcap C^2(Q)$ $(Q = \{(x,t) \mid 0 < x < 1,\ t > 0\})$ is a solution to:

$$\begin{cases} u_t(x,t) = a u_{xx}(x,t) + F(x,t), 0 < x < 1,\ t > 0,\ a > 0 \\ u(x,0) = f(x),\ 0 < x < 1, \\ u(0,t) = 0,\ t \geq 0, \\ u(L,t) = 0,\ t \geq 0. \end{cases} \tag{1}$$

(a) State and prove a version of the maximum principle.

(b) Assume the solution is given by $u(x,t) = \displaystyle\int_0^1 g(x,y,t)f(y)dy$. In the case that $F(x,t) \equiv 0$, show that $g(x,y,t) = \displaystyle\int_0^1 g(x,z,t-s)g(z,y,s)dz$ for $t > s > 0$.

(c) State and prove a version of the uniqueness of solutions to (1).

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2. Fourier Series.

(a) Show explicitly a Fourier series and an open interval $S = (a,b)$ such that the series converges pointwise in S but does not converge uniformly in S.

(b) State the Weirstrass approximation theorem with any assumptions necessary.

(c) Suppose $f(x)$ is a continuous 2π periodic function. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(2\pi n\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx \tag{2}$$

for any irrational α.

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3. Method of Characteristics.

Solve $(t^2+1)u_t(t,x) + x u_x(t,x) = u$, with the initial condition $u(0,x) = e^x$. (Solve all problems in terms of the original VARIABLES!)

TURN OVER

4. Wave equation.

Consider the forced wave equation

$$u_{tt} = u_{xx} + e^{-x}, \qquad t > 0, \qquad 0 \le x \le L. \tag{3}$$

with initial conditions $u(x,0) = g(x), u_t(x,0) = 0$, and boundary conditions $u(0,t) = u(L,t) = 0$.

(a) Find a formal solution in terms of the function g.

(b) Find conditions on g that guarantee that the expression you found in (a) is a solution of the system.

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5. Laplace's Equation

Let $B = B_a(0) \subset R^2, a > 0$. Consider the following boundary value problem:

$$\begin{cases} \Delta u = 0 \text{ in } B \\ u(a,\theta) = f(\theta) \; 0 < \theta \le 2\pi. \end{cases} \tag{4}$$

(a) Find a formal solution $u(r,\theta)$ that 'solves' the above boundary value problem.

(b) Find the conditions on f that assure that the formal solution u is continuous on $\overline{B}$. Give a proof of your conclusion.

(c) Find the conditions on f so that $u_r(r,\theta)$ (the partial derivative in r) is continuous in B.

(d) Show that there is a unique solution under the condition that you have found in (b) (you can use the fact that the formal solution you found is C^∞ in B as a function of x and y i.e. $w(x,y) = u(r(x,y), \theta(x,y))$ is continuously differentiable in x and y in any order.)