1. Let \( u(x, t) \) solve the following PDE

\[
    u_t + (G(u))_x = 0, \quad \text{on} \quad |x| < \infty, \quad t \geq 0,
\]

with \( G(u) = u(1 - u/u_0) \) and \( u_0 > 0 \).

Consider the following two cases for initial data:

Case \( \alpha \)

\[
    u(x, 0) = \begin{cases} 
        \frac{1}{2}u_0 & \text{for} \quad x < 0 \\
        u_0 & \text{for} \quad x > 0 
    \end{cases}
\]

Case \( \beta \)

\[
    u(x, 0) = \begin{cases} 
        u_0 & \text{for} \quad x < 0 \\
        0 & \text{for} \quad x > 0 
    \end{cases}
\]

For each case:

(a) (i) Solve the solution.

(ii) Sketch the characteristics. Annotate the solution along the characteristics.

(iii) Does the solution exist everywhere in the plane \( |x| < \infty, \quad t \geq 0 \).

(b) Discuss the important differences between case \( \alpha \) and \( \beta \).

2. Suppose that \( f(x) \) is a \( 2\pi \) periodic function and its Fourier coefficients, \( a_n \) and \( b_n \), exist. Let \( F(x) = \int_0^x (f(y) - \frac{1}{2}a_0) dy \).

(a) Find the Fourier series for \( F(x) \).

(b) Prove that the series you found in converges to \( F(x) \) for each \( x \). You may use the Riemann-Lebesgue lemma if you choose, providing that you clearly state it.

(c) Suppose \( G(x) = \sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n} \). Prove that the series representing \( G \) is NOT its Fourier series. (Hint: (b) should prove useful).
3. The normal deflection $w$ of an elastic membrane with uniform tension $\tau > 0$ and loading pressure $p(x, y)$ obeys the equation

$$\nabla^2 w = -\frac{1}{\tau} p(x, y).$$

(a) Suppose that the membrane is held fixed on its boundaries $x = 0, x = a, y = 0$ and $y = b$. Find the formal solution for the deflection in the domain $0 < x < a, 0 < y < b$.

(b) State conditions on the function $p$ so that your formal solution in (a) is a true solution.

4. Given the wave equation in $N$ dimensions

$$\nabla^2 u - \frac{1}{2} u_{tt} = 0$$

with

$$u(t_0, x) = f(x), \quad u_t(t_0, x) = g(x)$$

where $x = (x_1, \ldots, x_N)$, $\nabla^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ and $|x_i| < \infty$.

(a) Let $N = 1$. Find the solution $u$ corresponding to $f(x) = 0, g(x) = \delta(x - x_0)$ where (here and below) $\delta(x)$ is the Dirac delta function; evaluate all integrals and sketch the solution.

(b) (i) Let $N = 3$, $u(0, x) = 0$, $u_t(0, x) = g(\rho)$ where $\rho^2 = \sum_{i=1}^{3} x_i^2$. Find the general solution.

   Hint: the transformation: $u = w/\rho$ is helpful.

(ii) Suppose $u_t(0, x) = \delta(x)$. Integrate over a sphere of radius $R$ to find an integral identity for $g(\rho)$. 

5. (a) Given the equation

$$u_t - u_{xx} - u = 0, \quad |x| < \infty, \quad t > 0$$

with $u(0, x) = \delta(x - x_0)$ and $u$ and its derivatives vanish as $x \to \infty$. Find the solution $u$, evaluate all integrals and discuss important aspects of the solution.

(b) Given the equation

$$u_t - \nabla^2 u - u = 0, \quad |x| < \infty, \quad t > 0$$

where $x = (x_1, \ldots, x_N), \nabla^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ and $|x_i| < \infty$, $u(t = 0, x) = f(x)$ and $u$ and its derivatives vanish at $\infty$.

Show the solution, assuming it exists, is unique.

(c) Given the equation

$$u_t - u_{xx} = 0, \quad |x| < \infty, \quad t > 0$$

(i) Find the most general "self-similar" solution of the form:

$$u = \frac{1}{\sqrt{2t}} \ F(\xi), \quad \xi = \frac{x}{\sqrt{2t}}$$

(ii) Find the solution $F$ that vanishes as $|x| \to \infty$

(iii) Explain whether this solution is related to any special solution of the heat equation.