1. Let $D$ be a region in $\mathbb{R}^n$, and suppose that $u$ satisfies the equation

$$\nabla^2 u = F(x), \quad x \in D$$

a) Suppose $D$ is bounded and that

$$\frac{\partial u}{\partial n} = h(x), \quad x \text{ on } \partial D$$

where $h$ is given on the closed boundary $\partial D$, and $n$ is the outward unit normal. Discuss the uniqueness of the solution.

b) Consider the same problem as in (a) setting $F = 1$ and $h(x) = 1$. Discuss the existence of a solution.

c) Now suppose that $D$ is the exterior to the unit disc in $\mathbb{R}^2$: $D = \{x^2 + y^2 > 1\}$, that $F = 1$, and that $u = 1$ on $x^2 + y^2 = 1$. Find the general solution and the bounded solution.

2. Consider the equation

$$u_{tt} - c^2 u_{xx} = H(x, t), \quad |x| < \infty, \quad t \geq 0.$$ 

a) Let $u(x, 0) = f(x), u_t(x, 0) = g(x)$ and suppose that $H = 0$. Formulate an integral, $I, I \geq 0$ that can be associated with the energy. Show $I$ is a constant of the motion. Find a second integral, call it $J$, that is also a constant of motion.

b) Let $u(x, 0) = f(x) = 0, u_t(x, 0) = g(x) = K = \text{constant}$ and suppose that $H = 0$. Find the solution.

c) Suppose $H(x, t) = \delta(x)e^{\omega t}$ where $\delta(x)$ is the Dirac delta function. Discuss possible solutions.
3. Consider

\[ x^2 u_x + y^2 u_y = u, \quad x > 0, \quad y > 0 \]

a) Find and sketch the characteristics for the above equation.

b) Solve the above equation with the initial condition \( u(1, y) = y^2 \) for \( y > 0 \).

c) Does the solution exist everywhere in the first quadrant? If not, where does it fail to exist?

d) What is \( \lim_{t \to \infty} u(t, t) \)? What is \( \lim_{(x, y) \to (0, 0)} u(x, y) \)?

4. (a) Find a formal series solution of the problem

\[
\begin{align*}
&u_{tt} - u_{xx} - u = 0, \quad 0 < x < \pi, \quad t > 0 \\
&u(x, 0) = 0, \quad 0 < x < \pi \\
&u_t(x, 0) = g(x), \quad 0 < x < \pi \\
&u_x(0, t) = u_x(\pi, t) = 0, \quad t > 0
\end{align*}
\]

Define all constants in your solution in terms of the Fourier coefficients of \( g \).

(b) Give some reasonable conditions on \( g \) so that your series solution in part (a) is a classical solution of (1).

(c) Find the limiting behavior of \( |u| \) at \( x = \frac{\pi}{2} \) as \( t \to \infty \), that is find \( \lim_{t \to +\infty} |u(\frac{\pi}{2}, t)| \), if it exists.

5. Consider the Fourier series on \([0, \pi]\) given by

\[ f(x) = \sum_{n=1}^{\infty} \frac{n}{1 + n^2} \sin(nx) \]

a) State Parseval’s relation for \( f \).

b) Is \( f \in L_2 \)? Is \( f \) continuous?

c) What is the Fourier series for \( F(x) = \int_{0}^{x} f(\xi)d\xi \)?

d) Does the Fourier series for \( F(x) \) converge to a continuous function?