APPM GRADUATE PRELIMINARY EXAMINATION
PARTIAL DIFFERENTIAL EQUATIONS – SOLUTIONS

Thursday August 24, 2017, 10AM –1PM

There are five problems. Solve any four of the five problems. Each problem is worth 25 points.

On the front of your bluebook please write: (1) your name and (2) a grading table. Please start each problem with a new page. Text books, notes, calculators are NOT permitted. A sheet of convenient formulae is provided.

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1. (**First order equations**)

(a) (18 points)
Solve the first-order initial value problem

$$\kappa e^x \frac{\partial u}{\partial x} + (t+1)\frac{\partial u}{\partial t} = \sigma u, \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = 2e^{-x},$$

where κ, σ are positive constants.

(b) (7 points)
Solve the reduced equation when the parameter $\kappa = 0$. Show that this solution agrees with the limit as $\kappa \to 0$ of the solution to the first part of the problem.

Solution:

(a) We use the method of characteristics, and solve the system of ODEs,

$$\frac{dt}{ds} = t+1,$$
$$\frac{dx}{ds} = \kappa e^x,$$
$$\frac{du}{ds} = \sigma u.$$

Its general solution is

$$\ln(t+1) = s, \qquad e^{-x_0} - e^{-x} = \kappa s, \qquad u = u_0 e^{\sigma s}.$$

The integration constants x_0 and u_0 are related by the conditions
$t(s=0) = 0$, $x(s=0) = x_0$ and $u(s=0) = u_0$, so that the initial condition implies $u_0 = 2e^{-x_0}$. Then we use the last condition to express

$$e^{-x} = u_0/2 - \kappa s = u_0/2 - \kappa \ln(t+1),$$

from where we substitute $u_0 = 2(e^{-x} + \kappa \ln(t+1))$ into the equation for u and get

$$u = u_0 e^{\sigma s} = 2(e^{-x} + \kappa \ln(t+1)) e^{\sigma \ln(t+1)}.$$

Thus, finally we obtain the solution $u(x,t)$ satisfying the given initial condition by construction,

$$u(x,t) = 2(e^{-x} + \kappa \ln(t+1))(t+1)^{\sigma}.$$

(b) When $\kappa = 0$, the PDE reduces to the ODE

$$(t+1)\frac{du}{dt} = \sigma u,$$

its general solution is

$$u = u_0(x) \cdot (t+1)^\sigma$$

with $u_0(x)$ independent of t and the initial condition implies $u_0(x) = 2e^{-x}$. One sees that the limit of the general PDE solution from first part of the problem is obtained by simply putting $\kappa = 0$ there, which is exactly the solution of the ODE,

$$u = 2e^{-x}(t+1)^\sigma.$$

2. (**Heat type equations**)

(a) (15 points)
Solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} + \frac{2\kappa}{x+x_0}\frac{\partial u}{\partial x} + \mu(t)u, \qquad 0 < x < L,\ t > 0,$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = u_0(x),$$

where $\kappa > 0$, $x_0 > 0$ are constants and the function $\mu(t)$ is a bounded integrable function with support on a finite interval of t-axis.
Hint: The transformation $w = \tilde{w}\exp(\int^x \nu(\xi)d\xi/2)$ is useful to remove the $\frac{\partial u}{\partial x}$ term.

(b) (10 points)
Consider the behavior of the solution for large positive time t: Does it converge to a limit as $t \to \infty$? Give the approximate form of the solution for large t. For large t find an approximation to $T > 0$ such that the solution $u(L/2, t+T)$ differs from $u(L/2, t)$ by a factor of 2 ?

Solution:

(a) We use separation of variables, look for a solution of the PDE in the form $u(x,t) = T(t)X(x)$. Then the PDE reads

$$T'(t)X(x) = \kappa T(t)X''(x) + \frac{2\kappa}{x+x_0}T(t)X'(x) + \mu(t)T(t)X(x),$$

which can be rewritten as

$$\frac{T'(t)}{T(t)} - \mu(t) = \frac{\kappa X''(x)}{X(x)} + \frac{2\kappa}{x+x_0}\frac{X'(x)}{X(x)},$$

therefore both left- and right-hand side must be a constant, call it λ. Then we solve

$$T'(t) - (\mu(t) + \lambda)T(t) = 0 \qquad (1)$$

and

$$\kappa X''(x) + \frac{2\kappa}{x + x_0} X'(x) - \lambda X(x) = 0. \tag{2}$$

The solution of t-part eq. (1) is

$$T(t) = T(0) e^{\lambda t + \int_0^t \mu(\tau) d\tau}.$$

Since $\mu(t)$ has finite support, in order to have solutions bounded in time t, we must have $\lambda \leqslant 0$.

To solve eq. (2), we first change the dependent variable as

$$X(x) = \tilde{X}(x) \exp\left(-\int^x \frac{d\xi}{\xi + x_0}\right) = \frac{\tilde{X}(x)}{x + x_0}.$$

Then

$$X'(x) = \frac{\tilde{X}'(x)}{x + x_0} - \frac{\tilde{X}(x)}{(x + x_0)^2}, \qquad X''(x) = \frac{\tilde{X}''(x)}{x + x_0} - \frac{2\tilde{X}'(x)}{(x + x_0)^2} + \frac{2\tilde{X}(x)}{(x + x_0)^3},$$

and, substituting into eq. (2) gives

$$\kappa \tilde{X}''(x) - \frac{2\kappa \tilde{X}'(x)}{x + x_0} + \frac{2\kappa \tilde{X}(x)}{(x + x_0)^2} + \frac{2\kappa \tilde{X}'(x)}{x + x_0} - \frac{2\kappa \tilde{X}(x)}{(x + x_0)^2} - \lambda \tilde{X}(x) = 0,$$

i.e.

$$\kappa \tilde{X}''(x) - \lambda \tilde{X}(x) = 0.$$

Since $\lambda \leqslant 0$, let $\lambda = -\kappa k^2$ for some real k. Then in general

$$\tilde{X}(x) = C_1 \cos kx + C_2 \sin kx$$

with some constants C_1 and C_2. Now we make $X(x)$ satisfy the boundary conditions (BC) $u(0, t) = u(L, t) = 0$. The BC at $x = 0$ yields $C_1 = 0$, while the BC at $x = L$ then implies

$$\sin kL = 0 \qquad \Longrightarrow \qquad k = \frac{\pi n}{L},\ n \in \mathbb{Z}.$$

Thus, to satisfy the initial condition (IC) $u(x, 0) = u_0(x)$, we look for the solution $u(x, t)$ in the form of the series

$$u(x, t) = \frac{e^{\int_0^t \mu(\tau) d\tau}}{x + x_0} \sum_{n=1}^{\infty} b_n e^{-\kappa \pi^2 n^2 t / L^2} \sin \frac{\pi n x}{L}. \tag{3}$$

The IC then reads

$$\sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{L} = (x + x_0) u_0(x)$$

for $0 < x < L$. Then the coefficients b_n are determined by

$$b_n == \frac{2}{L} \int_0^L (x + x_0) u_0(x) \sin \frac{\pi n x}{L} dx.$$

With these b_n the solution is given by the series eq. (3).

(b) As $t \to \infty$, the solution $u(x,t) \to 0$. Since $\mu(t)$ is nonzero only on a finite interval of t, the approximate solution for large t can be written as

$$u(x,t) \approx \frac{e^{\int_0^\infty \mu(\tau)\,d\tau}}{x + x_0} \cdot b_1 e^{-\kappa\pi^2 t/L^2} \sin\frac{\pi x}{L},$$

i.e. it is determined by the lowest mode $k_1 = \pi/L$. The characteristic time of convergence to zero is $\sim L^2/(\pi^2\kappa)$ and the time T is determined by $u(L/2, t+T) = u(L/2,t)/2$, i.e.

$$T \approx \frac{L^2}{\pi^2\kappa}\ln 2.$$

3. (**Fourier series**)

(a) (10 pts)
Show that the pointwise convergent series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{1/2}}$$

cannot converge uniformly to a square integrable function f in $[-\pi,\pi)$.

(b) (15 pts)
Let $f(x)$ be 2π periodic and piecewise smooth. Prove that its Fourier series converges uniformly and absolutely to f.

Solution:

(a) Suppose the series converged uniformly to a square integrable function f. The Fourier coefficients of f are

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{m=1}^{\infty}\frac{\sin(mx)}{m^{1/2}}\sin(nx)\,dx. \tag{1}$$

Using the assumed uniform convergence, one can integrate term by term to obtain

$$b_n = \sum_{m=1}^{\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(mx)}{m^{1/2}}\sin(nx)\,dx = \frac{1}{n^{1/2}}. \tag{2}$$

Similarly, $a_n = 0$. Since f is assumed to be square integrable, Bessel's inequality would apply, and so

$$\infty = \sum_{n=1}^{\infty}\frac{1}{n} = \sum_{n=1}^{\infty} b_n^2 \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx < \infty, \tag{3}$$

a contradiction.

(b) See book, page 69.

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4. (**Wave type equations**)

Consider

$$u_{tt} - c^2 u_{xx} + a u_t + \frac{a^2}{4} u = 0\,, \qquad 0 \leqslant x \leqslant L\,,\ \ t > 0\,, \tag{4}$$
$$u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)\,, \quad u(0,t) = u(L,t) = 0\,,$$

where $f(x), g(x)$ are integrable and $c > 0$ and $a > 0$ are constants.

(a) (15 points)
Solve the above initial boundary value problem.
Hint: Look for solutions of the form $u(x,t) = e^{-\frac{a}{2}t} w(x,t)$.

(b) (5 points)
Derive the energy relation

$$\frac{dE}{dt} = -2a \int_0^L u_t^2 dx\,, \tag{5}$$
$$E(t) = \int_0^L \left[u_t^2 + u_x^2 + \frac{a^2}{4} u^2 \right] dx\,.$$

What physical effect do the additional terms $a u_t$ and $a^2 u/4$ in (4) represent?

(c) (5 points)
Using energy relation (5), prove that the solution found in part **(a)** is unique.

Solution:

(a) Substituting $u(x,t) = e^{-\frac{a}{2}t} w(x,t)$ into (4) gives $w_{tt} - c^2 w_{xx} = 0$.
Using separation of variables $w(x,t) = X(x)T(t)$, gives $T''(t)/T(t) = c^2 X''(x)/X(x) = k^2$, where $k^2 \geqslant 0$. Then, standard methods give the solution

$$u(x,t) = e^{-\frac{a}{2}t} \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{L}\right) \left[A_n \cos\left(\frac{\pi n c t}{L}\right) + B_n \sin\left(\frac{\pi n c t}{L}\right) \right]$$

The Fourier coefficients are defined by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx$$

and

$$B_n = \frac{L}{\pi n c} \left(\frac{2}{L} \int_0^L \left[g(x) + \frac{a}{2} f(x) \right] \sin\left(\frac{\pi n x}{L}\right) dx \right)$$

(b) Multiply (4) by u_t and integrate over the interval $[0, L]$. This gives

$$\int_0^L \frac{1}{2}(u_t^2)_t - c^2 u_t u_{xx} + a u_t^2 + \frac{a^2}{8}(u^2)_t dx = 0.$$

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The boundary conditions $u(0,t) = u(L,t) = 0$ imply $u_t(0,t) = u_t(L,t) = 0$. Performing integration-by-parts on the second term and applying these boundary conditions yields the desired energy relation

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left(u_t^2 + u_x^2 + \frac{a^2}{4}u^2\right) dx = -a\int_0^L u_t^2 dx.$$

The energy $E(t)$ is non-increasing in time, i.e. $E(t_2) \leqslant E(t_1)$ for $t_2 > t_1$, indicating some dissipative force (e.g. friction, vibration) is modeled by the terms au_t and $a^2u/4$.

(c) Suppose (4) has two distinct solutions: $u_1(x,t)$ and $u_2(x,t)$. Define $\tilde{u} \equiv u_1 - u_2$, which satisfies the equation $\tilde{u}_{tt} - c^2\tilde{u}_{xx} + a\tilde{u}_t + \frac{a^2}{4}\tilde{u} = 0$ and initial conditions $\tilde{u}(x,0) = 0, \tilde{u}_x(x,0) = 0, \tilde{u}_t(x,0) = 0$. As a result, energy relation (5) satisfies $0 \leqslant E(t) \leqslant E(0) = 0$. This implies that $E(t) = 0$ for all $t > 0$, or $E(t) = \int_0^L \left[\tilde{u}_t^2 + \tilde{u}_x^2 + \frac{a^2}{4}\tilde{u}^2\right] dx = 0$. Since $\tilde{u}$ is smooth, this means that $\tilde{u}_t^2 + \tilde{u}_x^2 + \frac{a^2}{4}\tilde{u}^2 = 0$. Since these are all non-negative quantities this implies that $\tilde{u}_t = \tilde{u}_x = \tilde{u} = 0$, or equivalently $u_1(x,t) = u_2(x,t)$.

5. (**Elliptic Equations**)

Let $u \in C^2(D) \cap C(\bar{D})$, where D is a smooth, bounded domain of $\mathbb{R}^n$.

(a) (10 points)
Show that there is at most one solution to the boundary value problem

$$\Delta u = \alpha u + h(x), \qquad x \in D,$$
$$\frac{\partial u}{\partial \nu} = \beta u + g(x), \qquad x \in \partial D,$$

where $\alpha > 0, \beta < 0$.

(b) (5 points)
State the Maximum-Minimum Principle.

(c) (10 pts)
Show that if $\Delta u \geqslant u$ in D and $u = f < 0$ on ∂D, then $u \leqslant 0$ in $\bar{D}$.

Solution:

(a) See book.

(b) Let's argue by contradiction. Suppose $u(x_0) = u_0 > 0$ for some $x_0 \in D$. Let H be the largest open connected subset of D containing x_0 such that $u > 0$ in H. (H can be defined as the union of all such open connected sets, which is nonempty since u is continuous.) Because $u = f < 0$ on ∂D, H must be contained in the interior of D, and by definition we must have $u = 0$ on ∂H: if $x \in \partial H$ and $u(x) > 0$, by continuity $u > 0$ in a neighborhood of x and H could be enlarged further, contradicting its definition. If $u(x) < 0$, then $u < 0$ in a neighborhood of x, and x could not have belonged to ∂H.

So we have a domain H such that $u > 0$ in H and $u = 0$ on ∂H. By the assumption, $\Delta u \geqslant u > 0$ in H. By the Maximum Principle, we must have $0 = \max_{\partial H} u \geqslant u(x_0) > 0$, a contradiction. Therefore, we must have $u \leqslant 0$ in D.

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