1. (First order equations)

(a) (18 points)
Solve the first-order initial value problem

\[ \kappa e^x \frac{\partial u}{\partial x} + (t+1) \frac{\partial u}{\partial t} = \sigma u, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u(x,0) = 2e^{-x}, \]

where \( \kappa, \sigma \) are positive constants.

(b) (7 points)
Solve the reduced equation when the parameter \( \kappa = 0 \). Show that this solution agrees with the limit as \( \kappa \to 0 \) of the solution to the first part of the problem.

2. (Heat type equations)

(a) (15 points)
Solve the initial boundary value problem

\[ \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \frac{2\kappa}{x+x_0} \frac{\partial u}{\partial x} + \mu(t)u, \quad 0 < x < L, \quad t > 0, \]

\[ u(0,t) = u(L,t) = 0, \]

\[ u(x,0) = u_0(x), \]

where \( \kappa > 0, x_0 > 0 \) are constants and the function \( \mu(t) \) is a bounded integrable function with support on a finite interval of \( t \)-axis.

*Hint:* The transformation \( w = \tilde{w}\exp(\int x^v(\xi) d\xi/2) \) is useful to remove the \( \frac{\partial u}{\partial x} \) term.

(b) (10 points)
Consider the behavior of the solution for large positive time \( t \): Does it converge to a limit as \( t \to \infty \)? Give the approximate form of the solution for large \( t \). For large \( t \) find an approximation to \( T > 0 \) such that the solution \( u(L/2, t+T) \) differs from \( u(L/2, t) \) by a factor of 2.

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3. (Fourier series)

(a) (10 pts)
Show that the pointwise convergent series
\[ \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{1/2}} \]
cannot converge uniformly to a square integrable function \( f \) in \((-\pi, \pi)\).

(b) (15 pts)
Let \( f(x) \) be \( 2\pi \) periodic and piecewise smooth. Prove that its Fourier series converges uniformly and absolutely.

4. (Wave type equations)
Consider
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + au_t + \frac{a^2}{4} u &= 0, & 0 \leq x \leq L, \ t > 0, \\
u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = u(L,t) = 0,
\end{align*}
\]
where \( f(x), g(x) \) are integrable and \( c > 0 \) and \( a > 0 \) are constants.

(a) (15 points)
Solve the above initial boundary value problem.
**Hint:** Look for solutions of the form \( u(x,t) = e^{-\frac{a}{2}t}w(x,t) \).

(b) (5 points)
Derive the energy relation
\[
\frac{dE}{dt} = -2a \int_0^L u_t^2 \, dx , \\
E(t) = \int_0^L \left[ u_t^2 + c^2 u_x^2 + \frac{a^2}{4} u^2 \right] \, dx .
\]
What physical effect do the additional terms \( au_t \) and \( \frac{a^2}{4}u \) in \(1\) represent?

(c) (5 points)
Using energy relation \(2\), prove that the solution found in part (a) is unique.

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5. (Elliptic Equations)

Let \( u \in C^2(D) \cap C(\bar{D}) \), where \( D \) is a smooth, bounded domain of \( \mathbb{R}^n \).

(a) (10 points)
Show that there is at most one solution to the boundary value problem

\[
\begin{align*}
\Delta u &= \alpha u + h(x), \quad x \in D, \\
\frac{\partial u}{\partial \nu} &= \beta u + g(x), \quad x \in \partial D,
\end{align*}
\]

where \( \alpha > 0, \beta < 0 \).

(b) (5 points)
State the Maximum-Minimum Principle.

(c) (10 pts)
Show that if \( \Delta u \geq u \) in \( D \) and \( u = f < 0 \) on \( \partial D \), then \( u \leq 0 \) in \( \bar{D} \).

Good Luck!!!