APPM GRADUATE PRELIMINARY EXAMINATION
PARTIAL DIFFERENTIAL EQUATIONS

Thursday August 24, 2017, 10AM –1PM

There are five problems. Solve any four of the five problems. Each problem is worth 25 points.

On the front of your bluebook please write: (1) your name and (2) a grading table. Please start each problem with a new page. Text books, notes, calculators are NOT permitted. A sheet of convenient formulae is provided.

---

1. (**First order equations**)

(a) (18 points)
Solve the first-order initial value problem

$$\kappa e^x \frac{\partial u}{\partial x} + (t+1)\frac{\partial u}{\partial t} = \sigma u, \qquad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = 2e^{-x},$$

where κ, σ are positive constants.

(b) (7 points)
Solve the reduced equation when the parameter $\kappa = 0$. Show that this solution agrees with the limit as $\kappa \to 0$ of the solution to the first part of the problem.

2. (**Heat type equations**)

(a) (15 points)
Solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \frac{2\kappa}{x + x_0}\frac{\partial u}{\partial x} + \mu(t) u, \qquad 0 < x < L,\ t > 0,$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = u_0(x),$$

where $\kappa > 0$, $x_0 > 0$ are constants and the function $\mu(t)$ is a bounded integrable function with support on a finite interval of t-axis.
Hint: The transformation $w = \tilde{w} \exp(\int^x \nu(\xi) d\xi/2)$ is useful to remove the $\frac{\partial u}{\partial x}$ term.

(b) (10 points)
Consider the behavior of the solution for large positive time t: Does it converge to a limit as $t \to \infty$? Give the approximate form of the solution for large t. For large t find an approximation to $T > 0$ such that the solution $u(L/2, t+T)$ differs from $u(L/2, t)$ by a factor of 2.

Continue to next page

APPM GRADUATE PRELIMINARY EXAMINATION
PARTIAL DIFFERENTIAL EQUATIONS

3. (**Fourier series**)

(a) (10 pts)
Show that the pointwise convergent series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{1/2}}$$

cannot converge uniformly to a square integrable function f in $[-\pi, \pi)$.

(b) (15 pts)
Let $f(x)$ be 2π periodic and piecewise smooth. Prove that its Fourier series converges uniformly and absolutely.

4. (**Wave type equations**)
Consider

$$u_{tt} - c^2 u_{xx} + a u_t + \frac{a^2}{4} u = 0\,, \qquad 0 \leqslant x \leqslant L\,,\ \ t > 0\,, \tag{1}$$
$$u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)\,, \quad u(0,t) = u(L,t) = 0\,,$$

where $f(x), g(x)$ are integrable and $c > 0$ and $a > 0$ are constants.

(a) (15 points)
Solve the above initial boundary value problem.
Hint: Look for solutions of the form $u(x,t) = e^{-\frac{a}{2}t} w(x,t)$.

(b) (5 points)
Derive the energy relation

$$\frac{dE}{dt} = -2a \int_0^L u_t^2 dx\,, \tag{2}$$
$$E(t) = \int_0^L \left[u_t^2 + c^2 u_x^2 + \frac{a^2}{4} u^2 \right] dx\,.$$

What physical effect do the additional terms $a u_t$ and $a^2 u/4$ in (1) represent?

(c) (5 points)
Using energy relation (2), prove that the solution found in part **(a)** is unique.

Continue to next page

APPM GRADUATE PRELIMINARY EXAMINATION
PARTIAL DIFFERENTIAL EQUATIONS

5. (Elliptic Equations)

Let $u \in C^2(D) \cap C(\bar{D})$, where D is a smooth, bounded domain of $\mathbb{R}^n$.

(a) (10 points)
Show that there is at most one solution to the boundary value problem

$$\begin{aligned} \Delta u &= \alpha u + h(x), && x \in D, \\ \frac{\partial u}{\partial \nu} &= \beta u + g(x), && x \in \partial D, \end{aligned}$$

where $\alpha > 0, \beta < 0$.

(b) (5 points)
State the Maximum-Minimum Principle.

(c) (10 pts)
Show that if $\Delta u \geqslant u$ in D and $u = f < 0$ on ∂D, then $u \leqslant 0$ in $\bar{D}$.

Good Luck!!!