Preliminary Examination (Solutions): Partial Differential Equations,
10:00 AM - 1:00 PM, Aug. 15, 2016,
Newton Lab.

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

Name:____________________________________

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points. A sheet of convenient formulae is provided.

1. **(Quasi-linear equations)** Consider the quasi-linear equation

$$u_y + uu_x = 1. \tag{1}$$

In each of the following parts, solve eq. (1) subject to the given data if possible. State the region in the x-y plane where the solution is classical and is unique. If it is not possible to obtain a unique, classical solution, state why.

(a) (10 pts) $u(x,x) = 2$.
(b) (8 pts) $u(y^2/4, y) = y/2$.
(c) (7 pts) $u(y^2/2, y) = y$.

Solution: Equation (1) can be solved by the method of characteristics. Given data along a curve $(x_0(\tau), y_0(\tau), u_0(\tau))$ for τ in some interval, we identify the characteristic equations

$$\begin{aligned}\frac{dx}{ds} &= u, \quad x(0,\tau) = x_0(\tau),\\ \frac{dy}{ds} &= 1, \quad y(0,\tau) = y_0(\tau),\\ \frac{du}{ds} &= 1, \quad u(0,\tau) = u_0(\tau).\end{aligned}$$

This initial value problem can be solved

$$\begin{aligned}x(s,\tau) &= \frac{1}{2}s^2 + u_0(\tau)s + x_0(\tau),\\ y(s,\tau) &= s + y_0(\tau),\\ u(s,\tau) &= s + u_0(\tau).\end{aligned}$$

In order to obtain a unique, classical solution to eq. (1), we must be able to effect the transformation $(x(s,\tau), y(s,\tau)) \to (s(x,y), \tau(x,y))$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} \end{vmatrix} = \begin{vmatrix} s + u_0(\tau) & 1 \\ u_0'(\tau)s + x_0'(\tau) & y_0'(\tau) \end{vmatrix}.$$

(a) The data is parameterized by

$$x_0(\tau) = \tau, \quad y_0(\tau) = \tau, \quad u_0(\tau) = 2, \quad \tau \in \mathbb{R}.$$

Then $J = s + 1$ so that the transformation $(x(s,\tau), y(s,\tau)) \to (s(x,y), \tau(x,y))$ is invertible near the data where $s = 0$. A calculation gives the unique solution

$$u(x,y) = 1 + \sqrt{1 + 4x - 4y},$$

valid so long as $y < x - 1/4$.

(b) The data is parameterized by

$$x_0(\tau) = \tau^2, \quad y_0(\tau) = 2\tau, \quad u_0(\tau) = \tau, \quad \tau \in \mathbb{R}.$$

Then $J = s$ so that the transformation $(x(s,\tau), y(s,\tau)) \to (s(x,y), \tau(x,y))$ is not guaranteed to be invertible near the data where $s = 0$. We can obtain a formal solution using the method of characteristics

$$x = \frac{1}{2}s^2 + \tau s + \tau^2, \quad y = s + 2\tau, \quad u = s + \tau,$$

which can be inverted to obtain the explicit form

$$u(x,y) = \frac{y}{2} \pm \frac{\sqrt{4x - y^2}}{2}.$$

For both signs of the radical, $u(y^2/4, y) = y/2$, so we see that the formal solution is not unique for the given data. Moreover, $u_x = \pm(4x - y^2)^{-1/2}$, so that the formal solution is not even differentiable at the data.

(c) The data is parameterized by

$$x_0(\tau) = \frac{\tau^2}{2}, \quad y_0(\tau) = \tau, \quad u_0(\tau) = \tau, \quad \tau \in \mathbb{R}.$$

Then $J = 0$ everywhere so that the transformation $(x(s,\tau), y(s,\tau)) \to (s(x,y), \tau(x,y))$ is not guaranteed to be invertible. The data actually lies on a characteristic curve because

$$\frac{dx_0}{d\tau} = \tau = u_0, \quad \frac{dy_0}{d\tau} = 1, \quad \frac{du_0}{d\tau} = 1.$$

This implies that there is no unique way to "propagate" the solution off the initial curve. The formal, characteristic solution is

$$x = \frac{(s+\tau)^2}{2}, \quad y = s + \tau, \quad u = s + \tau,$$

which depends on only one parameter $s + \tau$. For example, $u(x,y) = y$ and $u(x,y) = -1 - \sqrt{1 + 2x + 2y}$ are solutions so that the solution is not unique.

2. **(Heat equation)** (25 pts) State and prove the (weak) maximum principle for continuous solutions of

$$u_t = a(x,t)u_{xx}, \quad x \in (0,1), \quad t \in (0,T], \quad T > 0, \tag{2}$$

where a is continuous and $a(x,t) \geq a_0 > 0$ for all $x \in [0,1]$, $t \geq 0$.

Solution: Theorem. Let $U_T = (0,1) \times (0,T]$ and $\Gamma_T = \overline{U_T} \setminus U_T$ be its parabolic boundary. If $u(x,t)$ is continous on $\overline{U_T}$, twice continously differentiable on U_T, and satisfies eq. (2), then

$$\max_{(x,t)\in\overline{U_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

Proof. Let $M = \max_{\Gamma_T} u(x,t)$. Let $v(x,t) = u(x,t) + \epsilon x^2$ where $\epsilon > 0$. Then

$$v_t - a(x,t)v_{xx} = u_t - a(x,t)u_{xx} - 2\epsilon < 0, \tag{3}$$

since $u_t - a(x,t)u_{xx} = 0$. Suppose v has a local maximum in U_T at (x_0,t_0) with $t_0 < T$. Then $v_t = 0 = v_x = 0$ and $v_{xx} \leq 0$ at (x_0,t_0). This implies $v_t - a(x,t)v_{xx} \geq 0$ at (x_0,t_0). But this contradicts eq. (3) so v cannot have a maximum in the interior of U_T.

Now suppose v has a maximum on the line $t = T$ at (x_1,T). Then $v_x = 0$, $v_{xx} \leq 0$, and $v_t \geq 0$ at (x_1,T). This implies $v_t - a(x,t)v_{xx} \geq 0$, again contradicting (3).

Therefore, by continuity, the maximum of v on $\overline{U_T}$ occurs on Γ_T. This implies

$$u(x,t) + \epsilon x^2 \leq \max_{(y,t)\in\Gamma_T} u(y,t) + \epsilon y^2 \leq \max_{\Gamma_T} u + \epsilon, \quad \text{for all } (x,t) \in \overline{U_T},$$

Then

$$u(x,t) \leq M + \epsilon(1 - x^2) \leq M + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $u(x,t) \leq M$ for all $(x,t) \in \overline{U_T}$.

3. **(Fourier series)**

(a) (8 points) State the Weierstrass approximation theorem with any assumptions necessary.

(b) (17 points) Suppose $f(x)$ is a continuous 2π periodic function. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

for any irrational α. [Hint: prove it for e^{inx} first and then use (a)].

Solution:

(a) See book.

(b) Let α be irrational. First prove the statement for $f(x) = e^{imx}$. For $m = 0$ both quantities are 1. For $m \neq 0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{imx}dx = \frac{1}{2m\pi i}e^{imx}\Big|_0^{2\pi} = 0,$$

and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi inm\alpha} = \lim_{N\to\infty} \frac{1}{N}\frac{e^{2\pi im(N+1)\alpha} - 1}{e^{2\pi im\alpha} - 1} = 0, \tag{4}$$

since the denominator is not zero since α is irrational, and the norm of the numerator is bounded.

Now, for the given f, invoke the Weierstrass approximation theorem. Given $\epsilon > 0$, we can find a trigonometric polynomial $T_\epsilon(x)$ such that $\|T_\epsilon - f\|_\infty < \epsilon$. Informally, the idea is that the sum of f is close to the sum of T_ϵ, the sum of T_ϵ is close to the integral of T_ϵ (by the first part), and the integral of T_ϵ is close to the integral of f. To make this precise, we use the triangle inequality. We have

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\right| =$$
$$\left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{N}\sum_{n=1}^{N} T_\epsilon(2\pi n\alpha)\right.$$
$$+\frac{1}{N}\sum_{n=1}^{N} T_\epsilon(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} T_\epsilon(x)dx$$
$$\left.+\frac{1}{2\pi}\int_0^{2\pi} T_\epsilon(x)dx - \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\right|$$

Using the triangle inequality, we get

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\right| \leq \\ \frac{1}{N}\sum_{n=1}^{N} \|f - T_\epsilon\|_\infty \\ + \left|\frac{1}{N}\sum_{n=1}^{N} T_\epsilon(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} T_\epsilon(x)dx\right| \\ + \frac{1}{2\pi}\int_0^{2\pi} \|T_\epsilon - f\|_\infty dx\end{aligned}$$

Using $\|T_\epsilon - f\|_\infty < \epsilon$ we get

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\right| \leq \\ 2\epsilon + \left|\frac{1}{N}\sum_{n=1}^{N} T_\epsilon(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} T_\epsilon(x)dx\right|\end{aligned}$$

Now, $T_\epsilon(x) = \sum_{m=-M_\epsilon}^{M_\epsilon} a_m e^{imx}$. We have $\frac{1}{2\pi}\int_0^{2\pi} T_\epsilon(x)dx = a_0$, which cancels with the corresponding term in the sum, leaving

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\right| \leq 2\epsilon + \sum_{m=1}^{M_\epsilon} |a_m| \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n m\alpha}\right| \\ + \sum_{m=-M_\epsilon}^{-1} |a_m| \left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n m\alpha}\right|.\end{aligned}$$

By (4), there is an N_ϵ, independent of $|m| \leq M_\epsilon$, such that $\left|\frac{1}{N_\epsilon}\sum_{n=1}^{N_\epsilon} e^{2\pi i n m\alpha}\right| < \epsilon/(|a_m| M_\epsilon)$ for all m if $N > N_\epsilon$. Therefore, for $N > N_\epsilon$

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(2\pi n\alpha) - \frac{1}{2\pi}\int_0^{2\pi} f(x)dx\right| \leq 4\epsilon,$$

which shows the desired limit.

4. **(Wave equation)** Suppose u is a smooth solution of the initial/boundary value problem

$$\begin{aligned} &u_{tt} - u_{xx} = f_x(x), \qquad t > 0, \qquad 0 < x < 1 \\ &u(x,0) = h(x), \qquad u_t(x,0) = g(x), \\ &u(0,t) = u(1,t) = 0, \end{aligned} \tag{5}$$

with f, h and g smooth and $h(0) = h(1) = 0$.

(a) (6 pts) Show that

$$\frac{d}{dt}\int_0^1 [u_x^2 + u_t^2]dx = 2\int_0^1 f_x(x)u_t dx.$$

(b) (7 pts) Show that smooth solutions to the initial value problem (5) are unique.

(c) (12 pts) Show that there is a constant K, depending only on f, g, and h_x, such that

$$\int_0^1 [u_x^2 + u_t^2]dx < K, \qquad t > 0.$$

(Hint: You might want to use the inequality $ab \geq -a^2 - b^2/4$.)

Solution:

(a) Multiplying the equation by u_t and integrating from 0 to 1 we get

$$\begin{aligned} \int_0^1 [u_{tt}u_t - u_{xx}u_t]dx &= \int_0^1 f_x(x)u_t dx, \\ \int_0^1 \left[\frac{1}{2}\frac{\partial}{\partial t}(u_t)^2 - u_{xx}u_t\right] dx &= \int_0^1 f_x(x)u_t dx, \end{aligned}$$

Integrating by parts,

$$\int_0^1 \left[\frac{1}{2}\frac{\partial}{\partial t}(u_t)^2 + u_x u_{tx}\right] dx - u_t u_x|_0^1 = \int_0^1 f_x(x)u_t dx,$$

Using the boundary condition $u_t(0,t) = u_t(1,t) = 0$ and recognizing the second term in the integral as $\frac{1}{2}\frac{\partial}{\partial t}(u_x)^2$ we get

$$\frac{d}{dt}\int_0^1 [u_x^2 + u_t^2]dx = 2\int_0^1 f_x(x)u_t dx. \tag{6}$$

(b) Suppose there are two such solutions u_1 and u_2. Then their difference $w = u_1 - u_2$ satisfies

$$\begin{aligned} &w_{tt} - w_{xx} = 0, \qquad t > 0, \qquad 0 < x < 1 \\ &w(x,0) = 0, \qquad w_t(x,0) = 0, \\ &w(0,t) = w(1,t) = 0. \end{aligned}$$

Since $f_x \equiv 0$ for this problem, $\frac{d}{dt}\int_0^1 [w_x^2 + w_t^2]dx = 0$ and the energy $E(t) = \frac{1}{2}\int_0^1 [w_x^2 + w_t^2]dx$ is constant. At $t = 0$, $E(0) = 0$ by the initial conditions and thus $E(t) = \frac{1}{2}\int_0^1 [w_x^2 + c^2 w_t^2]dx = 0$ for all $t > 0$. This implies, since w is smooth, that $w_x \equiv 0$, $w_t \equiv 0$ in $[0, 1]$ and therefore $w(x, t) = W$, a constant that must be 0 by the initial conditions. Therefore $u_1 \equiv u_2$.

(c) Integrating (6) in time from 0 to t, letting $E(t) = \frac{1}{2}\int_0^1 [u_x^2 + u_t^2]dx$, we obtain

$$E(t) - E(0) = E(t) - \frac{1}{2}\int_0^1 [h_x(x)^2 + g(x)]dx = \int_0^1 f_x(x)(u(x,t) - h(x))dx.$$

Integrating by parts on the right hand side,

$$E(t) - \frac{1}{2}\int_0^1 [h_x^2 + g]dx = f(x)(u(x,t) - h(x))\Big|_0^1 - \int_0^1 f(x)(u_x(x,t) - h_x(x))dx.$$

On the boundary, $u = h = 0$, so

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 [h_x^2 + g]dx - \int_0^1 f(x)(u_x(x,t) - h_x(x))dx, \\ &= \frac{1}{2}\int_0^1 [h_x^2 + g]dx + \int_0^1 f h_x dx - \int_0^1 f(x) u_x(x,t)dx, \end{aligned}$$

Using the inequality $-ab \le a^2 + b^2/4$ for the last integral we get

$$E(t) \le \frac{1}{2}\int_0^1 [h_x^2 + g]dx + \int_0^1 f h_x dx + \int_0^1 f^2 dx + \frac{1}{4}\int_0^1 u_x(x,t)^2 dx.$$

Recalling $E(t) = \frac{1}{2}\int_0^1 [u_x^2 + u_t^2]dx$ and rearranging,

$$\int_0^1 \left[u_x^2 + 2u_t^2\right] dx \le 2\int_0^1 [h_x^2 + g]dx + 4\int_0^1 f h_x dx + 4\int_0^1 f^2 dx \equiv K,$$

and so

$$\int_0^1 \left[u_x^2 + u_t^2\right] dx \le \int_0^1 \left[u_x^2 + 2u_t^2\right] dx \le K.$$

5. **(Elliptic problem)**

(a) (10 pts) Consider the boundary value problem in the upper half plane

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad y > 0,$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}.$$

Construct Green's function for this Dirichlet problem.

(b) (5 pts) Show that if $v(x,y)$ is harmonic, so is $u(x,y) = v(x^2 - y^2, 2xy)$.

(c) (5 pts) Show that the transformation $(x,y) \to (x^2 - y^2, 2xy)$ maps the first quadrant onto the upper half-plane. *Hint: Use polar coordinates.*

(d) (5 pts) Consider the boundary value problem in the quarter plane

$$u_{xx} + u_{yy} = 0, \quad x > 0, \quad y > 0,$$
$$u(x,0) = f(x), \quad x > 0,$$
$$u(0,y) = g(y), \quad y > 0.$$

Construct Green's function for this Dirichlet problem using your results from (a)-(c).

Solution:

(a) The fundamental solution for Laplace's equation in $\mathbb{R}^2$ is

$$\Phi(r) = -\frac{1}{2\pi} \ln(r).$$

Using the method of images, the Green's function for the upper half plane is

$$G(\mathbf{x}, \mathbf{x}') = \Phi(|\mathbf{x} - \mathbf{x}'|) - \Phi(|\mathbf{x} - \mathbf{x}'^*|) = \frac{1}{2\pi} \ln\left(\frac{|\mathbf{x} - \mathbf{x}'^*|}{|\mathbf{x} - \mathbf{x}'|}\right)$$
$$= \frac{1}{4\pi} \ln\left(\frac{(x - x')^2 + (y + y')^2}{(x - x')^2 + (y - y')^2}\right),$$

where $\mathbf{y}'^* = (x', -y')$ is the reflection of $\mathbf{y}' = (x', y')$ about the line $y' = 0$.

(b) Let $\eta = x^2 - y^2$, $\xi = 2xy$. Suppose $v_{\eta\eta} + v_{\xi\xi} = 0$ then

$$u_x = 2xv_\eta + 2yv_\xi,$$
$$u_{xx} = 2v_\eta + 4x^2 v_{\eta\eta} + 8xyv_{\eta\xi} + 4y^2 v_{\xi\xi},$$
$$u_y = -2yv_\eta + 2xv_\xi,$$
$$u_{yy} = -2v_\eta + 4y^2 v_{\eta\eta} - 8xyv_{\eta\xi} + 4x^2 v_{\xi\xi}.$$

Combining these results

$$u_{xx} + u_{yy} = 0,$$

so $u(x,y)$ is harmonic.

(c) Suppose $x = r\cos\theta$, $y = r\sin\theta$ for $r > 0$, $0 < \theta < \pi/2$ so that (x, y) is in the first quadrant. Then $(\eta, \xi) = (r^2\cos 2\theta, r^2\sin 2\theta)$ necessarily lie in the upper half plane $\xi > 0$. Because the transformation is invertible, the converse is also true.

(d) Combining the previous results, the Green's function for the quarter plane is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\ln\left(\frac{(x^2 - y^2 - x'^2 + y'^2)^2 + (2xy + 2x'y')^2}{(x^2 - y^2 - x'^2 + y'^2)^2 + (2xy - 2x'y')^2}\right)$$

The same result can be obtained by the method of images with

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\ln\left(\frac{[(x - x')^2 + (y + y')^2][(x + x')^2 + (y - y')^2]}{[(x - x')^2 + (y - y')^2][(x + x')^2 + (y + y')^2]}\right).$$