PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
August 2013

You have three hours to complete this exam. Each problem is worth 25 points. Work only four of the five problems problems. Please mark which four that you choose—only four will be graded. Please start each problem on a new page. A sheet of convenient formulae is attached.

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1. Consider
$$x^2 u_x + y^2 u_y = u, \qquad x > 0,\ y > 0. \tag{1}$$
   (a) Find and sketch the characteristics for the above equation.
   (b) Solve the above equation with the initial condition $u(1, y) = y^2$ for $y > 0$.
   (c) Does the solution exist everywhere in the first quadrant? If not, where does it fail to exist?
   (d) What is $\lim_{t\to\infty} u(t, t)$? What is $\lim_{(x,y)\to(0,0)} u(x, y)$?

2. The one-dimensional wave equation
$$u_{tt} - c^2 u_{xx} = 0 \tag{2}$$
is known to have D'Alembert's solution
$$u(x, t) = F(x + ct) + G(x - ct). \tag{3}$$
   (a) Given the initial-value problem
$$u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \tag{4}$$
   deduce D'Alembert's solution formula for (2) using (3).
   (b) Solve the IVP:
$$u_{tt} - u_{xx} = 0 \quad \text{with} \quad u(x, 0) = x^3, \quad u_t(x, 0) = \sin x \tag{5}$$
   (c) Let
$$u(x, t) = \int_0^t U(x, t - s, s)ds, \tag{6}$$
   where $U(x, 0, s) = 0$ and $U_t(x, 0, s) = f(x, s)$ for $x \in R$.
   Find the conditions of $U(x, t - s, s)$ for $u(x, t)$ to be a solution to
$$u_{tt} - c^2 u_{xx} = f(x, t). \tag{7}$$

3. Let $u(r,t)$ satisfy the heat equation in three dimensions with radial symmetry,

$$\partial_t u = \frac{\kappa}{r^2}\partial_r\left(r^2\partial_r u\right), \qquad 0 < r < R, t > 0, \tag{8}$$

subject to boundary conditions,

$$u(0,t) \text{ bounded}, \qquad u(R,t) = T_0, \quad \text{both for all } t > 0, \tag{9}$$

and initial conditions,

$$u(r,0) = T_i, \quad 0 \le r < R. \tag{10}$$

(a) Simplify the problem by setting $u(r,t) = v(r,t)/r + T_0$. State the problem for $v(r,t)$ in complete detail.

(b) Solve the problem in (a) to find $v(r,t)$ explicitly.

(c) Find an explicit formula for $u(0,t)$. Does your representation of $u(0,t)$ converge for all $t > 0$? Does it converge absolutely for all $t > 0$? Does $u(0,t)$ ever change sign? For each question, justify your answer.

(d) Based on your results in (c), sketch the graph of $u(0,t)$ for $t \ge 0$. Justify your answer.

4. (a) Given the function $F(x) = x$. Find the Fourier sine series on $(0, L)$.

(b) State Parseval's identity along with any necessary assumptions and use this identity to find the sum $\sum_1^\infty \frac{1}{n^2}$. What can be said about the derivative of this Fourier series?

c) Suppose $g(x)$ is a 2π-periodic function, continuous on $[-\pi, \pi]$ with a Fourier series given by

$$g(x) = \sum_{-\infty}^{\infty} c_n e^{inx}.$$

Let $f(x)$ be a 2π-periodic function satisfying the differential equation

$$f''(x) + \Lambda f(x) = g(x), \quad \Lambda \ne n^2, \quad n = 0, \pm 1, \pm 2, ...$$

Find the Fourier series of $f(x)$ and prove that it converges everywhere.

5. Let D be a region in R^n, and suppose that u satisfies the equation

$$\nabla^2 u = F(x), \qquad x \in D. \tag{11}$$

(a) Suppose D is bounded and that

$$\frac{\partial u}{\partial n} = h(x), \qquad \forall x \text{ on } \partial D \tag{12}$$

where h is given on the closed boundary ∂D, and n is the outward unit normal. Discuss the uniqueness of the solution.

(b) Consider the same problem as in (a) setting $F(x) = 1$ and $h(x) = 1$. Discuss the existence of a solution.

(c) Now suppose that D is the exterior to the unit disc in $R^2 : D = \{x^2 + y^2 > 1\}$, that $F(x) = 1$, and that $u = 1$ on $x^2 + y^2 = 1$. Find the general solution.

(d) Is your solution to (c) bounded? If yes, state the conditions for boundedness.