PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
Aug. 18 2005

Instructions:

You have three hours to complete this exam. Each problem is worth 25 points. Work only four of the five problems problems. Please mark which four the you choose—only four will be graded. Please start each problem on a new page. A sheet of convenient formulae is attached.

1. a) Let \( u(x) \) satisfy

\[
\frac{u}{t} = \nabla^2 u, \quad x \in \mathbb{R}^n,
\]

\[
u(x, 0) = \delta(x) \quad u(x, t) \to 0, \quad \text{as} \quad |x| \to \infty
\]

where \( \delta(x) \) is the Dirac delta function which obeys: \( \int f(x')\delta(x - x')dx' = f(x) \)

Find an explicit representation of this solution; i.e. evaluate all integrals.

b) Call the solution in part a): \( K(x) \). If the initial condition is given as

\[
u(x, 0) = F(x)
\]

write the solution in terms of an integral involving \( K \) and \( F \).

c) Suppose \( u(x) \) satisfies

\[
u_t + \vec{V} \cdot \nabla u = \nabla^2 u, \quad x \in \mathbb{R}^n
\]

\[
u(x, 0) = f(x)
\]

where \( \vec{V} \) is a constant vector in n-dimensions. Find a formula which expresses how the solution in part b) must be modified.
2. Let $u(x, t)$ solve the following “forced Burgers” equation

$$\partial_t u + u \partial_x u = f(u), \quad -\infty < x < \infty, \quad t \geq 0$$

with initial data

$$u(x, 0) = \begin{cases} 
1 & \text{for } x \leq 0 \\
1 - x & \text{for } 0 < x < 1 \\
0 & \text{for } x \geq 1 
\end{cases}$$

Consider the following two cases

$$f(u) = \begin{cases} 
+u & \text{Case } \alpha \\
-u & \text{Case } \beta 
\end{cases}$$

a) Solve the above for cases $\alpha$ and $\beta$.

b) Show that a shock, i.e a discontinuity, develops in finite time $t = t_* < \infty$ for case $\alpha$ only.

3. Consider

$$\nabla^2 u = F(x), \quad x \in D$$

where $D$ is an bounded region in $\mathbb{R}^n$.

a) Suppose

$$\frac{\partial u}{\partial n} + a(x)u = h(x), \quad x \text{ on } \partial D$$

where $h$ given on the closed connected boundary $\partial D$, $a(x) > 0$ and $n$ is the outward unit normal. Prove that the solution is unique.

b) Suppose $\frac{\partial u}{\partial n} = g(x)$ on $\partial D$. Find a necessary condition in terms of $F, g$ alone (not $u$) for the solution to exist.

c) Suppose $D$ is the disk $r < 2$ in $\mathbb{R}^2$, $F = 0$ and $u = 3\sin 2\theta + 1$ on $r = 2$. Without solving the equation

i) Find the maximum value of $u$ for $r \leq 2$.

ii) Find $u(0)$. 

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4. a) Given the function $F(x) = x$. Find the Fourier sine series for $F$ on $(0, L)$. What can be said about the derivative of this Fourier series? Prove the Parseval identity for the appropriate class of functions and use this identity to find the sum $\sum_{1}^{\infty} \frac{1}{n^2}$.

b) Suppose $g(x)$ is a $2\pi$-periodic function, continuous on $[-\pi, \pi]$ with a Fourier series given by

$$g(x) = \sum_{-\infty}^{\infty} c_n e^{inx}.$$ 

Let $f(x)$ be a $2\pi$-periodic function satisfying the differential equation

$$f''(x) + \Lambda f(x) = g(x), \quad \Lambda \neq n^2, \quad n = 0, \pm 1, \pm 2, \ldots$$

Find the Fourier series of $f(x)$ and prove that it converges everywhere.

5. Consider the equation

$$u_{tt} - u_{xx} = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad |x| < \infty \quad \text{(W)}$$

a) Suppose that

$$f(x) = 0, \quad g(x) = \{\text{sign}(x), |x| < 1; 0, |x| > 1\}$$

Solve for $u(x, t)$ and sketch. Note: $\text{sign}(x) = \{1, x > 0; -1, x < 0\}$.

b) Now consider the boundary conditions:

$$f(x) = \{\text{sign}(x)h(x), h(0) = 0\}, \quad g(x) = 0$$

Solve for the solution. Discuss the significance of this boundary condition.

c) Suppose we are given the equation

$$u_{tt} - u_{xx} + u = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad |x| < \infty \quad \text{(K)}$$

where $f, g$ are absolutely integrable. Solve this problem. Discuss any significant differences between the solution to problems (W) and (K).
Interesting formulae

1. Trig. identities

\[
\begin{align*}
\sin \alpha &= \frac{e^{i\alpha} - e^{-i\alpha}}{2i}, \quad \cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \\
\sinh \alpha &= \frac{e^{\alpha} - e^{-\alpha}}{2}, \quad \cosh \alpha = \frac{e^{\alpha} + e^{-\alpha}}{2} \\
\cos^2(\theta) + \sin^2(\theta) &= 1 \\
\cosh^2(\theta) - \sinh^2(\theta) &= 1
\end{align*}
\]

2 sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta),
2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta),
\sin(2\theta) = 2\sin(\theta)\cos(\theta)
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)
\sinh(2\theta) = 2 \sinh(\theta) \cosh(\theta)
\cosh(2\theta) = \cosh^2(\theta) + \sinh^2(\theta)

2. Indefinite Integrals

\[
\begin{align*}
\int \sin x \, dx &= -\cos x + C, \\
\int x \sin(ax) \, dx &= -\frac{1}{a} x \cos(ax) + \frac{1}{a^2} \sin(ax) \\
\int x^n \sin(ax) \, dx &= -\frac{1}{a} x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) \, dx, \quad n > 0 \\
\int x^n \cos(ax) \, dx &= \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx, \quad n > 0 \\
\int \sin ax \cos bx \, dx &= -\frac{1}{2} \left[ \frac{\cos(a-b)x}{a-b} + \frac{\cos(a+b)x}{a+b} \right], \quad a \neq b^2 \\
\int \cos ax \cos bx \, dx &= \frac{1}{2} \left[ \frac{\sin(a-b)x}{a-b} + \frac{\sin(a+b)x}{a+b} \right], \quad a \neq b^2 \\
\int \sin ax \sin bx \, dx &= \frac{1}{2} \left[ \frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b} \right], \quad a \neq b^2 \\
\int (\cos ax)^2 \, dx &= \frac{1}{2} \left[ x + \frac{\sin(2ax)}{2a} \right] \\
\int (\sin ax)^2 \, dx &= \frac{1}{2} \left[ x - \frac{\sin(2ax)}{2a} \right] \\
\int e^{ax} \sin bx \, dx &= \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx] \\
\int e^{ax} \cos bx \, dx &= \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \\
\int x^n e^{ax} \, dx &= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx
\end{align*}
\]

3. Definite Integrals

\[
\begin{align*}
\int_0^\infty e^{-ax^2} \, dx &= \frac{1}{2} \sqrt{\frac{\pi}{a}}, \quad \text{Re}(a) \geq 0 \\
\int_0^\infty x^2 e^{-ax^2} \, dx &= \frac{1}{4} \sqrt{\frac{\pi}{a^3}}, \quad \text{Re}(a) > 0 \\
\int_0^\infty \cos(px) e^{-ax^2} \, dx &= \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{p^2}{4a}}, \quad \text{Re}(a) \geq 0 \\
\int_0^\infty \frac{\sin x}{x} \, dx &= \frac{\pi}{2}
\end{align*}
\]
4. Formal Fourier Facts

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}. \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(z) \cos \frac{n\pi z}{L} \, dz, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(z) \sin \frac{n\pi z}{L} \, dz. \]

\[ \frac{1}{L} \int_{-L}^{L} |f(z)| \, dz = \frac{a_0}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2. \]

\[ \hat{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{i\omega t} \, dt, \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega)e^{-i\omega t} \, d\omega \]

5. Laplacian operator

\[ \nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \]

\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \phi^2} \]

6. Div, Grad, Curl and all that

\[ \nabla (a \cdot b) = a \cdot \nabla b + b \cdot \nabla a + a \times (\nabla \times b) + b \times (\nabla \times a) \]

\[ \nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b) \]

\[ \nabla \times (a \times b) = a (\nabla \cdot b) - b (\nabla \cdot a) + (b \cdot \nabla) a - (a \cdot \nabla) b \]

\[ \int_D \nabla \cdot F \, dx = \int_{\partial D} n \cdot F \, ds \]

\[ \int_D \nabla \times A \cdot nds = \oint_{\partial D} A \cdot dl \]

\[ \int_D (v \nabla^2 u + \nabla v \cdot \nabla u) \, dx = \int_{\partial D} v (n \cdot \nabla u) \, ds \]

\[ \int_D (v \nabla^2 u + u \nabla v) \, dx = \int_{\partial D} (v \nabla u - u \nabla v) \, ds \]

\[ \int_D \nabla^2 u \, dx = \int_{\partial D} n \cdot \nabla u \, ds \]