Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
January 2014

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

1. **Root Finding.**
   Construct a continuous function $f(x)$, defined over $x \in (-\infty, \infty)$ such that, for any starting point x_0 that is not itself a root, the Newton iterations for solving $f(x) = 0$ will be uniquely defined, stay bounded, but nevertheless fail to converge.

2. **Numerical quadrature.**
   The trapezoidal rule has error $O(h^2)$ and Simpson's rule error $O(h^4)$, in both cases with even powers only in their full error expansions. These are the first two members of the Newton-Cotes family of methods, with errors (starting from the trapezoidal case) h raised to 2, 4, 4, 6, 6, 8, 8, ...

   (a) Show that Simpson's rule can be obtained by a one step Richardson extrapolation of the trapezoidal rule.

   (b) Determine the quadrature weights in the scheme that is obtained by a one step Richardson extrapolation of Simpson's rule. Explain whether this is another member of the Newton-Cotes sequence. If this is the case, determine also if there will be any further instances of one-step Richardson extrapolation of a Newton-Cotes method giving another Newton-Cotes method.

3. **Interpolation/Approximation.**
   The *barycentric form* of Lagrange's interpolation polynomial takes the form

$$p_n(x) = \frac{\sum_{j=0}^{n} w_j\, f(x_j)/(x - x_j)}{\sum_{j=0}^{n} w_j/(x - x_j)},$$

   where $w_j = 1/\Psi_n'(x_j)$ with $\Psi_n(x) = \prod_{j=0}^{n}(x - x_j)$. Verify that the expression above indeed produces the (unique) interpolation polynomial.

4. **Linear Algebra**
Describe steps of QR algorithm to compute eigenvalues of a matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$.

(a) Describe and explain the purpose of the reduction to Hessenberg form.

(b) Describe the steps of QR iteration.

(c) Show that a sequence of QR iterates is a sequence of matrices similar to $\mathbf{A}$ by a unitary transformation.

(d) State a sufficient condition for the convergence of QR iteration.

5. **ODEs**
Consider a system of ODEs

$$\mathbf{y}' = \mathbf{f}\,(t, \mathbf{y})\,, \quad \mathbf{y}(0) = \mathbf{y}_0,$$

and the so-called Heun's method (a simple Runge-Kutta scheme),

$$\begin{aligned} \mathbf{k}_1 &= h\,\mathbf{f}\,(t_n, \mathbf{y}_n)\,, \\ \mathbf{k}_2 &= h\,\mathbf{f}\,(t_n + h, \mathbf{y}_n + \mathbf{k}_1)\,, \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2}\,(\mathbf{k}_1 + \mathbf{k}_2) \end{aligned}$$

(a) Determine the order of this method.

(b) How do you find if the method is stable?

(c) Define and determine the region of absolute stability for this method method.

6. **PDEs.**

(a) Formulate a Fourier-type method to solve the Poisson equation in the unit square $[0, 1] \times [0, 1]$,

$$\Delta u = f,$$

with the Dirichlet boundary conditions on three sides and the Neumann condition on the forth,

$$\begin{aligned} u\,(x, y)_{|x=0} &= 0, \\ u\,(x, y)_{|x=1} &= 0, \\ u\,(x, y)_{|y=0} &= 0, \\ \tfrac{\partial u}{\partial y}\,(x, y)_{|y=1} &= 0. \end{aligned}$$

For simplicity assume that the function f satisfies the same boundary conditions and is smooth together with a sufficient number of its partial derivatives.

(b) Formulate the corresponding discrete problem.

(c) Describe a fast algorithm for solving this equation.