Numerical Analysis Preliminary Exam

January 17, 2012

Time: 180 Minutes

Do 4 and only 4 of the following 6 problems. Please indicate clearly which 4 you wish to have graded.

!!! No Calculators Allowed !!!

!!!Show all of your work !!!

NAME:______________________________________

For Grader Only	
1	/ 25
2	/ 25
3	/ 25
4	/ 25
5	/ 25
6	/ 25
Σ	/100

1. **Nonlinear Equations** Given scalar equation, $f(x) = 0$,

    1. Describe I) Newtons Method, II) Secant Method for approximating the solution.
    2. State sufficient conditions for Newton and Secant to converge. If satisfied, at what rate will each converge?
    3. Sketch the proof of convergence for Newton's Method.
    4. Write Newton's Method as a fixed point iteration. State sufficient conditions for a general fixed point iteration to converge.
    5. Apply the criterion for fixed point iteration to Newton's Method and develop an alternate proof for Newton's Method.

Numerical quadrature:

2. Assume that a quadrature rule, when discretizing with n nodes, possesses an error expansion of the form

$$I - I_n = \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \ldots$$

Assume also that we, for a certain value of n, have numerically evaluated I_n , I_{2n} and I_{3n}.

a. Derive the best approximation that you can for the true value I of the integral.

b. The error in this approximation will be of the form $O(n^{-p})$ for a certain value of p. What is this value for p?

Interpolation / Approximation:

3. The *General Hermite interpolation problem* amounts to finding a polynomial $p(x)$ of degree $\alpha_1 + \alpha_2 + \ldots + \alpha_n - 1$ that satisfies

$$
\begin{aligned}
p^{(i)}(x_1) &= y_1^{(i)}, \quad i = 0, 1, \ldots, \alpha_1 - 1 \\
&\vdots \\
p^{(i)}(x_n) &= y_n^{(i)}, \quad i = 0, 1, \ldots, \alpha_n - 1,
\end{aligned}
$$

where the superscripts denotes derivatives, that is, we specify the first $\alpha_j - 1$ derivatives at the point x_j, for $j = 1, 2, \ldots, n$. Show that this problem has a unique solution whenever the x_i are distinct.

Hint: Set up the linear system for a small problem, recognize the pattern, and prove the general result.

4. **Linear Algebra**

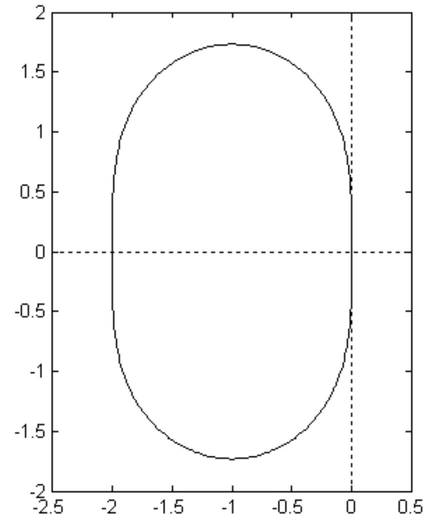
Consider the $n \times n$, nonsingular matrix, A. The Frobenius norm of A is given by

$$\|A\|_F = (\sum_{i,j} |a_{i,j}|^2)^{1/2}$$

1. Construct the perturbation, ∂A, with smallest Frobenius norm such that $A - \partial A$ is singular. (Hint: use one of the primary decompositions of A.)
2. What is the Frobenius norm of this special ∂A?
3. Prove that it is the smallest such perturbation.
4. Extra Credit: Is it unique?

Numerical ODE:

5. Consider using forward Euler (same as AB1; Adams-Bashforth of first order) as a predictor, and the trapezoidal rule (same as AM2; Adams Moulton of second order) as a corrector for solving the ODE $y' = f(t, y)$.



a. Write down the explicit steps that need to be taken in order to advance the numerical solution from time t_n to time $t_{n+1} = t_n + k$.

b. Determine the order of the combined scheme. In case you know a theorem that gives the order directly, you may quote this *in its general form*, i.e. do not just state the answer in the present special case.

c. The figure to the right illustrates the stability domain of the scheme. Prove that $(-2, 0)$ is the leftmost point in the domain, and that its vertical extremes are taken at $(-1 \pm \sqrt{3}\, i)$.

Note: If your solution utilizes that the stability domain is symmetric around the line $\text{Re}\,\xi = -1$, that symmetry has also to be proved.

6. **Partial Differential Equations**

Consider the steady-state, advection-diffusion equation in one space dimension:

$$-\partial_x(a(x)\partial_x u(x)) + b(x)\partial_x u = f, \qquad x \in [0,1]$$

with boundary conditions $u(0) = u(1) = 0$ and the assumption that $a(x)$ is continuous and $a(x) > 0$ for $x \in [0,1]$

1. Describe the finite difference (FD) method for approximating the solution using I) Centered Differences, II) Upwind Differences on the advection term. Let h represent the mesh spacing and assume a uniform mesh. In each case above, describe the linear systems, A_c^h and A_u^h, that the FD method yields.
2. Assume $a > 0$ and $b > 0$ are constant. State a relationship between a, b, and h that assures the eigenvalues of the linear system are real for I) Centered Differences, A_c^h, and II) Upwind Differences, A_u^h.
3. For constant $a > 0$, $b > 0$, use Gershgorin bounds to establish bounds on the eigenvalues of A_u^h, the **upwind** difference matrix.

Now consider the parabolic equation (assume $a > 0$ and $b > 0$ are constant)

$$\partial_t u = a\partial_{xx}u(x) - b\partial_x u, \qquad x \in [0,1]$$

(d) Write the **Forward** Euler scheme for this equation using I) Centered Differences II) Upwind Differences for the advection term.

(e) Write a simple relationship, in terms of a, b, h and δt that guarantees the stability of **Forward** Euler and **Upwind** Differences.