1. Root finding.
   
a) Describe Newton’s method for finding a root of \( f : [a, b] \to \mathbb{R} \).

b) State and prove a theorem about quadratic convergence of the method. Be as general as you can and be sure to include whatever assumptions you need on the derivatives of \( f \) and the initial guess.

**Solution:** See formula (2.2.1) on page 58 (don’t forget that Newton’s method assumes that an initial guess, \( x_0 \), is given) and Theorem 2.1 on pages 60-61 of Atkinson.

2. Orthogonal polynomials. Consider \( C[-1, 1] \) under the weighted inner product defined by \( (f, g) \equiv \int_{-1}^{1} \frac{fg}{\sqrt{1-x^2}} dx \).
   
a) What three-term recursion formula generates the corresponding orthogonal Chebyshev polynomials, \( T_k \)?

b) Remember that \( T_0 = 1 \) and \( T_1 = x \). Compute \( T_2 \) and the Chebyshev polynomial of degree 2 that best approximates \( f(x) = 4x^3 \). (You can avoid integrals by determining \( T_3 \) and writing \( f \) as a linear combination of the \( T_k \), \( 0 \leq k \leq 3 \).)

c) Describe a quadrature rule, \( I_n(f) \), for approximating \( I(f) \equiv \int_{-1}^{1} \frac{f}{\sqrt{1-x^2}} dx \) based on \( T_0, T_1, \cdots, T_n \) for general \( n \geq 0 \) and \( f \in C[-1, 1] \). Do not evaluate anything in this rule–just describe its form in terms of the \( T_k \).

d) Compute the error, \( I(f) - I_2(f) \), for the case \( f(x) = 4x^3 \)?

**Solution:**

(a) \( T_{n+1} = 2xT_n(x) - T_{n-1}(x) \).

(b) \( T_0 = 1, T_1 = x, T_2 = 2x^2 - 1 \). Since \( f = 4x^3 - 3x + 3x = T_3 + 3T_1 \), then the best Chebyshev approximation must be \( C_2 = 3T_1 = 3x \) (the minimum of \( \|T_3 + p_2\|_2 \) over all polynomials of degree two is just \( \|T_3\|_2 \), so the minimum of \( \|f - C_2\|_2 = \|T_3 + 3T_1 - C_2\|_2 \) is attained by \( C_2 = 3T_1 = 3x \).

(c) The quadrature formula would be \( I_n(f) = I(C_n) = \sum_{k=0}^{n} \frac{f(T_k)}{(T_k, T_k)} \int_{-1}^{1} \frac{T_k(x)}{\sqrt{1-x^2}} dx \).

(d) \( I(f) - I_2(f) = I(f - C_2) = \int_{-1}^{1} \frac{4x^3 - 3x}{\sqrt{1-x^2}} dx = 0 \) because the integrand is odd.
3. Linear algebra.

a) Define the concept of a vector norm on \( \mathbb{R}^n \).

b) Is \( \|x\| \equiv \sup_{p \geq 1} (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} \) a vector norm on \( \mathbb{R}^n \)? (You may use the fact that \( \|x\|_p \equiv (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} \) is a norm.)

c) Is \( \|x\| \equiv \lim_{p \to \infty} (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} \) a vector norm on \( \mathbb{R}^n \)?

d) Suppose that some vector norm, \( \| \cdot \| \), on \( \mathbb{R}^n \) is given. Define the concept of a vector induced matrix norm (or operator norm) on \( \mathbb{R}^{n \times n} \) and prove that it satisfies \( \|Ax\| \leq \|A\| \|x\| \) and \( \|AB\| \leq \|A\| \cdot \|B\| \) for any \( x \in \mathbb{R}^n \) and any \( A, B \in \mathbb{R}^{n \times n} \).

Solution:

a) \( \| \cdot \| \) is a vector norm if:

i. \( \|x\| \geq 0 \quad \forall x \neq 0 \) and \( \|x\| = 0 \) implies \( x = 0 \).

ii. \( \|\alpha x\| = |\alpha|\|x\| \quad \forall x \in \mathbb{R}^n \) and \( \forall \alpha \in \mathbb{R} \),

iii. \( \|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n \).

b) Yes. To see this, note for fixed \( p \) that \( \|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}}\|x\|_\infty \). Taking the sup over \( p \) thus yields \( \|x\|_\infty \leq \|x\| \leq n\|x\|_\infty \). This proves that the first condition holds. The second and third are true because \( \|x\|_p \) is a norm.

c) Yes, again because \( \|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}}\|x\|_\infty \), which upon taking limits shows that \( \|x\| = \|x\|_\infty \).

d) \( \|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \). But then \( \frac{\|Ax\|}{\|x\|} \leq \|A\| \) for all \( x \neq 0 \), which implies that \( \|Ax\| \leq \|A\| \|x\| \). The case \( x = 0 \) is obvious. We can use this to also conclude that \( \|AB\| = \sup_{x \neq 0} \frac{\|ABx\|}{\|x\|} \leq \|A\| \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} = \|A\| \cdot \|B\| \).

4. Tridiagonal matrix properties. Define the \( k \times k \) tridiagonal matrix

\[
T_k = \begin{bmatrix}
  a_1 & b_2 & & \\
  c_2 & a_2 & b_3 & \\
  & c_3 & a_3 & \ddots \\
  & & \ddots & \ddots & b_k \\
  & & & c_k & a_k \\
\end{bmatrix}.
\]

The characteristic polynomial of \( T_k \) is given by \( p_k(\lambda) = \text{det}(\lambda I - T_k) \).

a) Define \( p_k(\lambda) \) in terms of \( p_{k-1}(\lambda) \) and \( p_{k-2}(\lambda) \).

b) Show that if \( c_j b_j > 0 \) for \( j = 2, \ldots, k \), then \( p_k(\lambda) = 0 \) has only real roots. (Hint: find a real similarity transformation that symmetrizes \( T_k \).)

c) Assume \( c_j b_j > 0 \) for \( j = 2, \ldots, k \) and assume that the roots of \( p_{k-2}(\lambda) \) separate the roots of \( p_{k-1}(\lambda) \), that is, between each adjacent pair of roots of \( p_{k-1}(\lambda) \), there is a root of \( p_{k-2}(\lambda) \). Prove that the roots of \( p_{k-1}(\lambda) \) separate the roots of \( p_k(\lambda) \). (Hint: draw a picture and use the recursion.)
Solution:

(a) Expanding the last column of \( \det(\lambda I - T_k) \) yields
\[
p_k(\lambda) = (\lambda - a_k)p_{k-1}(\lambda) - b_kc_kp_{k-2}(\lambda).
\]

(b) Let \( r_i = \sqrt{b_j/c_j}, d_1 = 1.0 \) and \( d_j = r_jd_{j-1} \) for \( j > 1 \). Define the matrix \( D_k = \text{diag}(d_1, \ldots, d_j, \ldots, d_k) \). This yields
\[
D_kT_kD_k^{-1} = \begin{bmatrix}
a_1 & \sqrt{b_2c_2} & & \\
\sqrt{b_2c_2} & a_2 & \sqrt{b_3c_3} & \\
& \sqrt{b_3c_3} & a_3 & \ddots \\
& & \ddots & \ddots & \sqrt{b_kc_k} \\
& & & \sqrt{b_kc_k} & a_k
\end{bmatrix}.
\]

(c) Denote the roots of \( p_\ell(\lambda) \) by \( \lambda_\ell^1 < \lambda_\ell^2, \ldots, < \lambda_\ell^\ell \). Using the recursion derived above we see that
\[
p_k(\lambda_j^{k-1}) = -b_kc_kp_{k-2}(\lambda_j^{k-1})
p_k(\lambda_{j+1}^{k-1}) = -b_kc_kp_{k-2}(\lambda_{j+1}^{k-1})
\]
Since the roots of \( p_{k-2}(\lambda) \) separate the roots of \( p_{k-1}(\lambda) \), we have \( p_{k-2}(\lambda_j^{k-1})p_{k-2}(\lambda_{j+1}^{k-1}) < 0 \) and conclude that \( p_k(\lambda) \) must have at least one root between each root of \( p_{k-1}(\lambda) \).

Since each \( p_\ell(\lambda) \) is monic and \( \lambda_{k-1}^{k-1} \) is greater than all the roots of \( p_{k-2}(\lambda) \), then we may conclude that \( p_{k-2}(\lambda_{k-1}^{k-1}) > 0 \). Consider the equation
\[
p_k(\lambda_{k-1}^{k-1}) = -b_kc_kp_{k-2}(\lambda_{k-1}^{k-1}) < 0.
\]
Since \( p_k(\lambda) \) is also monic, this implies that \( p_k(\lambda) \) has a root greater than \( \lambda_{k-1}^{k-1} \). A similar argument shows the \( p_k(\lambda) \) has a root less than \( \lambda_1^{k-1} \).

Thus, the roots of \( p_{k-1}(\lambda) \) separate the roots of \( p_k(\lambda) \).

5. Ordinary differential equations. Consider the two step method (Adams-Bashforth)
\[
y_{n+2} = y_{n+1} + h \left[ \frac{3}{2} f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n) \right]
\]
Show that it is convergent and find its order. Sketch is region of absolute stability. State the relevant theorems.

Solution: The characteristic equation is
\[
r^2 - (1 + \frac{3}{2}h\lambda)r + \frac{1}{2}h\lambda = 0
\]
See Atkinson 6.8 for details.
Consider the heat equation
\[ \frac{\partial \phi}{\partial t} = \partial_x (a(x) \partial_x \phi), \]
with initial condition
\[ \phi|_{t=0} = \phi_0, \]
and periodic boundary conditions on the interval \([0, 1]\). Fully describe the Crank-Nicolson scheme for this problem, using a staggered grid for the spatial operator. Taking \(a(x) = 1\), show that the scheme is unconditionally stable.

Solution: We have
\[ \frac{d\phi_j}{dt} = \left[ a_{j-\frac{1}{2}} \phi_{j-1} - (a_{j-\frac{1}{2}} + a_{j+\frac{1}{2}})\phi_j + a_{j+\frac{1}{2}} \phi_{j+1} \right] / h_x^2, \]
where \(\phi_j = \phi(h \cdot j)\), \(a_{j-\frac{1}{2}} = a(h \cdot (j - \frac{1}{2}))\), \(h = 1/N\), and \(j = 1, \ldots, N\). Periodicity is imposed by setting \(\phi_0 = \phi_{N+1}\). Writing this system of ODEs as
\[ \frac{d\phi}{dt} = \frac{1}{h_x^2} A\phi, \]
where \(\phi = \{\phi_1, \ldots, \phi_N\}^t\) and \(A\) is a tridiagonal matrix, we apply the trapezoidal rule in time as follows:
\[ \phi^{k+1} - \phi^k = \frac{1}{2} \frac{h_t}{h_x^2} A(\phi^{k+1} + \phi^k) \]
where \(\phi^0 = \phi_0\). If \(a(x) = 1\), then matrix \(A\) is circulant, it is diagonalized by the Discrete Fourier transform, and it has non-positive eigenvalues. We have
\[ \phi^{k+1} = (I - \frac{1}{2} \frac{h_t}{h_x^2} A)^{-1} (I + \frac{1}{2} \frac{h_t}{h_x^2} A) \phi^k \]
Since \(A\) is non-positive definite, it is easy to verify that the eigenvalues of \((I - \frac{1}{2} \frac{h_t}{h_x^2} A)^{-1}(I + \frac{1}{2} \frac{h_t}{h_x^2} A)\) are all less or equal to 1, confirming unconditional stability.