Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August, 2017

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

1. **Root Finding.** Newton's method applied to the equation $f(x) \equiv x^3 - x = 0$ takes the form of the iteration
$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1}, \quad n = 0, 1, 2, \ldots$$
   (a) Study the behavior of the iteration when $x_0 > 1/\sqrt{3}$ to conclude that the sequence $\{x_0, x_1, \ldots\}$ approaches the same root as long as you choose $x_0 > 1/\sqrt{3}$.
   (b) Assume $-\alpha < x_0 < \alpha$. For what number α does the sequence always approach 0?
   (c) For an arbitrary $f(x)$, suppose that $f'(x) f''(x) \neq 0$ in an interval $[a, b]$, where $f''(x)$ is continuous and $f(a) f(b) < 0$. Show that if $f(x_0) f''(x_0) > 0$, for $x_0 \in [a, b]$, then the sequence $\{x_0, x_1, x_2, \ldots\}$ generated by Newton's method converges monotonically to a root $\alpha \in [a, b]$.

2. **Numerical quadrature.** Stirling's formula approximating $n!$ for large n states that:

   For every $n \geq 2$ there is a C_n such that $n! = C_n \sqrt{n} \, (n/e)^n$.

   Naturally, this is only useful if there is some extra information about C_n, which is usually provided as part of Stirling's formula. One way to obtain such information is as follows. First note that
$$\ln(n!) = \sum_{k=1}^{n} \ln(k) = \frac{1}{2} \sum_{k=1}^{n-1} (\ln(k) + \ln(k+1)) + \frac{1}{2} \ln(n).$$
   Next note that the sum in the rightmost expression is the trapezoid-rule approximation to the integral
$$\int_1^n \ln(x) \, \mathrm{d}x = n \ln(n) - n + 1$$
   using $n-1$ equal steps. Use the standard (not Euler-MacLaurin!) formula for the error in the trapezoid rule to obtain the following bounds on C_n in Stirling's formula:
$$e^{1 - \frac{\pi^2}{72}} \leq C_n \leq e^{1 - \frac{1}{12}}.$$
   You may find the following formula useful: $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$.

3. **Interpolation/Approximation.** Let $\{x_1, x_2, \ldots, x_n\}$ be n real and distinct interpolation nodes and let V be the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

   (a) Let $l_i(x)$, $i = 1, \ldots, n$ be the Lagrange interpolating polynomials. Use the fact $l_i(x_k) = \delta_{ik}$ to show that V is non-singular.

   (b) Let

$$\Phi_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n) = \sum_{j=1}^{n+1} a_j x^{j-1}.$$

   Outline an algorithm for finding the entries of V^{-1} based on finding the coefficients of $l_i(x)$, $i = 1, \ldots, n$. Hint: relate $l_i(x)$ to $q_i(x) = \Phi_n(x) / (x - x_i)$.

4. **Linear Algebra** Consider the equations

$$w_0 = w_N; \qquad w_1 = w_{N+1}; \qquad w_{j+1} + 4w_j + w_{j-1} = f_j, \text{ for } j = 1, \ldots, N.$$

   (a) Show that the resulting system $A\mathbf{w} = \mathbf{f}$ for $\mathbf{w} = [w_1, \ldots, w_N]^T$ can be written as

$$(A' + C)\mathbf{w} = \mathbf{f},$$

   where A' is a tridiagonal matrix and C is a rank one (or rank 2) matrix (your choice).

   (b) Describe a method for solving the system which is $O(N)$ in terms of cost and storage.

5. **ODEs**

   (a) Construct the absolute stability regions in the complex plane for the explicit and implicit Euler methods.

   (b) Consider the initial value problem $\dot{u} = -u^3$, $u(0) = u_0$ which has the solution

$$u(t) = \operatorname{sign}(u_0)/\sqrt{2t + u_0^{-2}}, \text{ or } u(t) = 0 \text{ for } u_0 = 0.$$

   The solution satisfies $\lim_{t\to\infty} u(t) = 0$ for any u_0. Find a condition on the step size h (which will depend on u_0) which guarantees that the solution obtained by applying the explicit Euler method to this problem will also satisfy $\lim_{n\to\infty} u_n = 0$, where u_n is the approximation to $u(nh)$.

   (c) The same as (b), but using the implicit Euler method. First find the range of step size h for which the solution of nonlinear equation exists, then find the range of h for which the solution satisfies $\lim_{n\to\infty} u_n = 0$.

   (d) Use Taylor series in the limit $h \to 0$ to obtain an explicit upper bound on the error in the implicit Euler approximation after the first time step of the problem from (c). What happens to the bound in the limit $h \to \infty$? What happens to the actual error in the first time step as $h \to \infty$?

6. **PDEs**

Determine coefficients c_{-1}, c_0, c_1 so that the scheme

$$u_j^{n+1} = c_{-1} u_{j-1}^n + c_0 u_j^n + c_1 u_{j+1}^n$$

for the solution of the equation

$$u_t + a u_x = 0, \quad a > 0,$$

agrees with the Taylor expansion of $u(x_j, t_{n+1})$ to as high an order as possible (the result is the Lax-Wendroff scheme).

(a) Derive a system of equations for the coefficients c_{-1}, c_0, c_1 (so that the approximation is second order accurate in space and time).

(b) Solve this system and write the answer in terms of $\nu = a\frac{h_t}{h_x}$, where h_t is the step size in time and h_x is the step size in space.

(c) Find the stability condition of the scheme.