Department of Applied Mathematics

Preliminary Examination in Numerical Analysis

August 17, 2016 , 10 am – 1 pm.

Submit solutions to four (and no more) of the following six problems. Show all your work, and justify all your answers. No calculators allowed.

1. Root finding / Nonlinear equations

Consider the scalar equation $F(x)=0$. Assume α is a root of the equation.

a. Give the recursion for the Newton method for approximating a root.

b. Give conditions on $F(x)$ near α that guarantee convergence for x_0 sufficiently close to α.

c. Consider the following Taylor expansion: $F(\alpha)=F(x)+F'(x)(\alpha-x)+\frac{1}{2}F''(w)(\alpha-x)^2$, where $w=\beta x+(1-\beta)\alpha$ for some $\beta\in[0,1]$. Using the Taylor expansion, derive a relationship between the error at step $j+1$ in terms of the error at step j.

d. Show how the conditions stated in part (b), together with the Taylor expansion above, can be used to bound the error at step $j+1$ in terms of the error in step j.

e. Finally, show how the development above can be used to establish convergence. With what order does the iteration converge?

2. Quadrature

Determine the nodes and the weights in the 2-node Gaussian quadrature formula

$$\int_0^\infty f(x)\,e^{-x}dx = w_1 f(x_1)+w_2 f(x_2).$$

3. Interpolation / Approximation

a. Define what is meant by *cubic splines* and, for these, *natural* and *not-a-knot* conditions.

b. Determine the *not-a-knot* cubic spline $s(x)$ that satisfies the data

x	-1	0	1	2
y	-2	-3	-4	1

c. If, at the nodes $x=-h,0,h$, one has function values y_{-h}, y_0, y_h and forms a quadratic interpolant $s(x)$, one obtains $s'(0)=[-\frac{1}{2}y_{-h}+\frac{1}{2}y_h]/h$, i.e. the finite difference weights can be written as $[-\frac{1}{2}, 0, +\frac{1}{2}]/h$. It might be tempting to replace the quadratic interpolant here with a natural cubic spline (hoping to increase the approximation's order of accuracy). Work out the weights you get in this case.

4. Linear Algebra

Consider the linear system $A\underline{x} = \underline{b}$, where $A_{n\times m}$, $\underline{x}_{m\times 1}$, $\underline{b}_{n\times 1}$.

a. Describe the three possible cases for existence and uniqueness of a solution of the linear system. Give criteria on $A, \underline{b}$ that distinguish each case.

b. Let $\underline{x}_{LS}$ be a minimizer of the least squares functional, that is, let

$$\| A\underline{x}_{LS} - \underline{b} \|_2 = \min_{\underline{x}} \| A\underline{x} - \underline{b} \|_2 .$$

   (i) Does $\underline{x}_{LS}$ always exist? Explain your answer.

   (ii) Give conditions on $A, \underline{b}$ such that $\underline{x}_{LS}$ is unique.

   (iii) In the case of a unique solution, give an expression for the least squares solution $\underline{x}_{LS}$.

   (iv) If there is an infinite number of solutions to the least squares problem, find the solution of minimal norm.

c. The minimal norm solution can be computed by using the singular value decomposition (SVD) of *A*. Define the singular value decomposition and show how it can be used to compute the minimal norm least squares solution.

5. Numerical ODE

The following are two different linear multistep methods for solving the ODE $y'(t) = f(t, y(t))$:

(i) $$y_{n+1} = y_n + \frac{k}{2}(3f(t_n, y_n) - f(t_{n-1}, y_{n-1}))$$

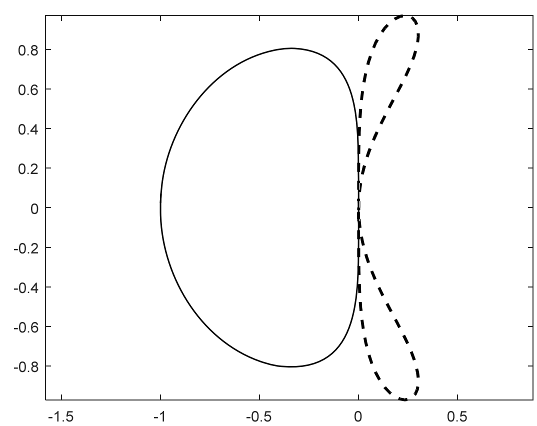
(ii) $$y_{n+1} = y_{n-1} + \frac{k}{3}(7f(t_n, y_n) - 2f(t_{n-1}, y_{-1}) + f(t_{n-2}, y_{n-2}))$$

In order to assess the basic properties of these two schemes, we run the Matlab code

```
r = exp(pi*2i*linspace(0,1));
plot(r.*(r-1)./((3*r-1)/2),'k-'); hold on
plot((r.^3-r)./((7*r.^2-2*r+1)/3),'k--');
axis equal
xi = 0.2+0.8i; roots([1,-7*xi/3,2*xi/3-1,-xi/3])
```

and obtain the plot shown to the right, together with the output

```
ans =
  -0.3082 + 1.1297i
   0.7431 + 0.9314i
   0.0318 - 0.1944i
```



For each of the two schemes, determine

a. Will the schemes convergence to the ODE solution in the limit of $k \to 0$, or diverge?

b. What is their formal order of accuracy?

c. Identify their stability domains,

d. Do the schemes feature A-stability?

<u>6. Numerical PDE</u>

Consider the parabolic equation

$$\frac{\partial u}{\partial t} = a\,\frac{\partial^2 u}{\partial x^2} + f$$

where a is a constant.

a. Give the formula for the following finite difference approximations.
   - (i) Forward Euler: Centered differences in space, forward difference in time.
   - (ii) Backward Euler: Centered differences in space, backward difference in time.
   - (iii) Leapfrog: Centered difference in space and centered difference in time.

b. What is the order of accuracy of each method?

c. Use a von Neumann analysis (or any appropriate analysis) to determine the stability of each method.