Department of Applied Mathematics

Preliminary Examination in Numerical Analysis

August 2015

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

1. Root finding

Formulate Newton's method for solving the nonlinear 2×2 system of equations

$$\begin{cases} f(x,y) &= 0 \\ g(x,y) &= 0 \end{cases}.$$

In the same style as how one proves quadratic convergence in the scalar case for $f(x)=0$, show quadratic convergence (assuming sufficient smoothness of *f*, *g*, root being simple, etc.) in the 2×2 case. Assuming the root $x=\alpha, y=\beta$ to be of multiplicity one, define $\varepsilon_n = x_n-\alpha,\ \eta_n = y_n-\beta$, and show that both ε_{n+1} and η_{n+1} are of size $O(\varepsilon_n^2,\eta_n^2)$.

2. Quadrature

Consider the quadrature formula

$$I_{quad} = \sum_{i=0}^{n} \alpha_i f(x_i), \quad x_i \in [-1,1] \tag{1}$$

for the integral

$$I = \int_{-1}^{1} f(x)w(x)dx ,$$

where $w(x)$ is a positive weight function in (-1,1). The (distinct) nodes $x_0, x_1, \ldots, x_n$ are chosen arbitrarily. The coefficients α_i are then chosen so the formula becomes exact for all polynomials of degree *n*. Let

$$\Omega_{n+1}(x) = \prod_{i=0}^{n}(x-x_i)$$

denote the polynomial of degree $n+1$ associated with the quadrature nodes. Prove that

$$\int_{-1}^{1} \Omega_{n+1}(x)p(x)w(x)dx = 0 \tag{2}$$

for any polynomial of degree less than or equal to $m-1$ if and only if the quadrature formula is exact for all polynomials of degree less than or equal to $n+m$.

3. Interpolation / Approximation

Assuming that $\varphi_n,\ n=0,1,2,\ldots$ form a set of orthogonal polynomials of degrees *n* over some interval [*a*,*b*] with weight function *w*(*x*) > 0, show that they obey a three-term recursion relation of the form

$$\varphi_{n+1}(x)-(a_n x+b_n)\varphi_n(x)+c_n\varphi_{n-1}(x)=0, \qquad n=1,2,3,\ldots$$

where the coefficients a_n, b_n, c_n do not depend on *x*.

4. Linear Algebra

Let $A\in\mathbb{C}^{n\times n}$ be a symmetric complex valued matrix, $A=A^T$. It is possible to show that one can find vectors u and nonnegative numbers μ solving the so-called con-eigenvalue problem

$$A\bar{u}=\mu u\ .$$

Show that these vectors u form an orthonormal basis and there exists a unitary matrix $U\in\mathbb{C}^{n\times n}$ and a nonnegative matrix

$$\Sigma=\operatorname{diag}(\mu_1,\mu_2,\ldots,\mu_n)$$

such that

$$A=U\Sigma U^T.$$

5. Numerical ODE

Consider the 4th order Adams-Bashforth scheme (AB4) for solving the ODE $y'=f(x,y)$:

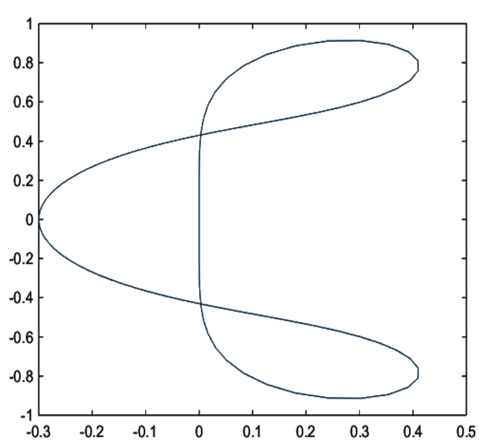
$$y_{n+1}=y_n+\frac{h}{24}\left(55f_n-59f_{n-1}+37f_{n-2}-9f_{n-3}\right)\ .$$

a. Apply the *root condition* to this scheme. Explain the outcome of the test, and explain what information this provides regarding the scheme.

b. The Matlab code

```
r  = exp(complex(0,linspace(0,2*pi)));
xi = 24*(r.^4-r.^3)./(55*r.^3-59*r.^2+37*r-9);
plot(xi);
```

generates the figure shown to the right.



i. Derive the relation used in the code.

ii. Explain (no need to do the algebra) how you would go from the figure to obtain the *stability domain* for the AB4 scheme.

iii. Explain what information a stability domain conveys.

6. Numerical PDE

Consider the Poisson's equation

$$(\partial_{xx} + \partial_{yy})u = f(x,y), \quad (x,y) \in B = [0,1]\times[0,1]$$

with the Dirichlet boundary condition

$$u\big|_{(x,y)\in \partial B} = 0.$$

Set f to be

$$\begin{aligned} f(x,y) \;=\; & -4\pi^2[\cos(2\pi x) - 4\cos(4\pi x)][\cos(2\pi y) - \cos(4\pi y)] \\ & -4\pi^2[\cos(2\pi y) - 4\cos(4\pi y)][\cos(2\pi x) - \cos(4\pi x)] \end{aligned}$$

yielding the solution

$$u(x,y) = (\cos(2\pi x) - \cos(4\pi x))(\cos(2\pi y) - \cos(4\pi y)).$$

At a first glance it may appear that seeking a solution as a sine series,

$$u(x,y) = \sum_{m,n=1}^{\infty} u_{mn} \sin(\pi m x)\sin(\pi n y)$$

should be an efficient approach. However, it turns out that the sine series converges rather slowly.

(a) Can you figure out why the convergence of the sine series is fairly slow?

(b) What are other bases one can use to achieve high accuracy? Suggest a basis that would be more efficient in this case.

(c) Sketch a numerical scheme to compute the solution with high accuracy.