Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August, 2013

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

1. **Root Finding.**

   (a) Show that Newton's method for the scalar equation $f(x) = 0$ is quadratically convergent (assuming that $f(x)$ is sufficiently smooth, and that $f'(x) \neq 0$ at the root).

   (b) In case that one can readily evaluate also $f''(x)$, the enhanced (cubically convergent) version

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}$$

   may be used. Derive this formula.

2. **Numerical quadrature.**

   (a) The trapezoidal rule takes the form

   $$\int_a^b f(x)\,dx = h\left[\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{n-2} + f_{n-1} + \frac{1}{2}f_n\right] + O(h^2),$$

   where the step size $h = [b-a]/n$. Show that it indeed is second order accurate.

   (b) In case $f(x)$ is periodic over $x \in [a, b]$ and furthermore infinitely differentiable, the error improves beyond any algebraic order. This can be shown by considering its Fourier expansion. In the case of $f(x) = e^{\cos x}$ over $x \in [0, 2\pi]$, one would thus consider the expansion $e^{\cos x} = \sum_{k=0}^{\infty} a_k \cos kx$, for which it can be shown that $\lim_{k\to\infty} a_k 2^{k-1} k! = 1$. Assuming (correctly) that this estimate is quite good also at finite k-values, provide a rough estimate for the trapezoidal error when applied to $\int_0^{2\pi} e^{\cos x} dx$ with $n = 6$.

3. **Interpolation/Approximation.**

   (a) Write down Lagrange's form of the polynomial $p_n(x)$ of degree n that interpolates function values f_k at node locations x_k, $k = 0, 1, \ldots, n$.

   (b) Prove that this interpolation polynomial is unique.

   (c) If we additionally are provided values for f'_k, determine what degree the corresponding Hermite interpolation polynomial will have, and describe **two** genuinely different procedures for creating this Hermite polynomial.

4. **Linear Algebra**

   Let A be an $n \times n$ antisymmetric matrix with real entries: $A^t = -A$.

   (a) Show that if n is odd then $\det(A) = 0$

   (b) Show that if n is even then $\det(A) \geq 0$

   (c) Assuming that A has only simple eigenvalues and applying the QR iteration based on orthogonal matrices, what will be the structure of the matrix (similar to A) in the limit of the QR iteration?

5. **ODEs**

   (a) Define the region of absolute stability (stability domain) for a general multistep method. How do you find if the method is stable? How do you determine the order of a multistep method? Why are these notions useful?

   (b) Prove that the region of absolute stability of an explicit multistep method is always bounded.

6. **PDEs**

   Consider Lax-Wendroff method for the wave equation $u_t + cu_x = 0$, $c > 0$, with the periodic boundary conditions $u(t, x + 2\pi) = u(t, x)$,

$$u_j^{n+1} - u_j^n = -\frac{1}{2}c\mu\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}(c\mu)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right),$$

   where $\mu = h_t/h_x$, $u_j^n = u(nh_t, jh_x)$, $h_x = 2\pi/N$, $j = 0, 1, \ldots N-1$, and $n = 1, 2, \ldots$.

   (a) Determine the order of the method

   (b) Determine the precise stability conditions (you may set $c = 1$ to simplify the derivation).