1. Nonlinear equations: Solution:

The function $f(x) = x - g(x)$ is continuous on $[a, b]$ and crosses the axis: $f(a) = a - g(a) < 0 < b - g(b) = f(b)$. Hence, there exists at least one zero, u, of f (that is, a fixed point of g) in $[a, b]$. Assume also that $g(v) = v \neq u$. Then $0 < |u - v| = |g(u) - g(v)| < \lambda|u - v| < |u - v|$, a contradiction. Thus, $u = v$ and we have proved uniqueness. Convergence holds as follows:

$$|u - x_{n+1}| = |g(u) - g(x_n)| \leq \lambda|u - x_n|,$$

which, by induction, implies convergence of x_n to u according to

$$|u - x_n| \leq \lambda^n |u - x_0|.$$

The explicit linear convergence bound now follows:

$$|x_{n+1} - u| = |g(x_n) - g(u)| \leq \lambda|x_n - u|.$$

2. **Numerical quadrature:** **Solution:**

We first note that symmetry tells that $\alpha = \gamma$. (If there were solutions with $\alpha \neq \gamma$, we would obtain equally valid ones with α and β interchanged, and averaging these formulas will also create valid formulas with the coefficients for $u(0)$ and $u(1)$ equal.)

In all the three cases, the resulting formula should be exact for the test function $u(x) = 1$, implying

$$1 = 2\alpha + \beta. \tag{1}$$

It thus only remains in each of the three cases to find a second test function, giving a second equation for the two unknowns.

a. Trapezoidal rule:

This quadrature formula should be exact for piecewise linear functions. Hence, consider for example

$$u(x) = \begin{cases} x & , \quad 0 \le x \le \frac{1}{2} \\ 1 - x & , \quad \frac{1}{2} \le x \le 1 \end{cases} .$$

It should now hold $\int_0^1 u(x)\,dx = \frac{1}{4} = \alpha \cdot 0 + \beta \cdot \frac{1}{2} + \alpha \cdot 0$. Together with (1), we obtain $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$.

b. Simpson's formula:

This method should be exact for an arbitrary quadratic function, in particular for $u(x) = x(1-x)$. We now get $\int_0^1 u(x)\,dx = \frac{1}{6} = \alpha \cdot 0 + \beta \cdot \frac{1}{4} + \alpha \cdot 0$, i.e. $\alpha = \frac{1}{6}$, $\beta = \frac{2}{3}$.

c. Natural spline:

In this case, it is natural to construct a second test function as follows: Let $u(x)$ over $0 \le x \le \frac{1}{2}$ be a cubic polynomial with the properties

$$u(0) = 0,\ \ u''(0) = 0,\ \ u(\tfrac{1}{2}) \neq 0,\ \ u'(\tfrac{1}{2}) = 0, \tag{2}$$

and then define $u(x)$ for $\frac{1}{2} \le x \le 1$ as the reflection around $x = \frac{1}{2}$, i.e. as $u(1-x)$. This function $u(x)$ is a natural cubic spline over $[0,1]$. It is straightforward to see that for ex. $u(x) = x - \frac{4}{3}x^3$ obeys the requirements (2), and satisfies $u(\frac{1}{2}) = \frac{1}{3}$, $\int_0^{1/2} u(x)\,dx = \frac{5}{48}$. We thus obtain as our second equation $\frac{5}{24} = \frac{1}{3}\beta$, and can conclude that $\alpha = \frac{3}{16}$, $\beta = \frac{5}{8}$.

3. **Interpolation Approximation:** **Solution:**

Since e is continuous, there must exist $\alpha, \beta \in [a,b]$ that satisfy

$$M = e(\alpha) = \max_{x \in [a,b]} e(x) \qquad \text{and} \qquad m = e(\beta) = \min_{x \in [a,b]} e(x).$$

Then the polynomial $p_n^* + (M+m)/2$ satisfies

$$-\frac{M-m}{2} = m - \frac{M+m}{2} \leq f(x) - [p_n^* + (M+m)/2] \leq M - \frac{M+m}{2} \leq \frac{M-m}{2}.$$

We must therefore have $M = (M-m)/2$, that is, $M = -m$, which proves the result.

4. **Linear algebra:** **Solution:**

(a) This is a result of the following identities:

$$\max_{x\neq 0}\frac{\|QARx\|^2}{\|x\|^2}=\max_{y\neq 0}\frac{\|QARR^*y\|^2}{\|R^*y\|^2}=\max_{y\neq 0}\frac{\|QAy\|^2}{\|y\|^2}=\max_{y\neq 0}\frac{< A^TQ^*QAy,y >}{< y,y >}=\max_{y\neq 0}\frac{< A^TAy,y >}{< y,y >}.$$

(b) $A=U\Lambda V^*$, where $U,V\in\Re^{n\times n}$ are unitary and Λ is $n\times n$ diagonal.
(c) $\|A\|=\|U\Lambda V^*\|=\|\Lambda\|=\|V\Lambda U^*\|=\|A^*\|=\|A^T\|$.
(d) Suppose $Au=\lambda u$, where $0\neq u\in\Re^n$ and $|\lambda|=\rho$. Then $\rho(A)=\|\lambda u\|/\|u\|=\|Au\|/\|u\|\leq\|A\|$.
(e) $\|A\|^2=max_{x\neq 0}< A^TAx,x > / < x,x >= max_{x\neq 0}< A^2x,x > / < x,x >=\rho(A^2)=\rho(A)^2$.
(f) The matrix $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ satisfies $0=\rho(A)<<\|A\|=1$:

5. **Numerical ODEs:** **Solution:**

a. The stencil for the scheme includes the stencil for Forward Euler, but has two additional entries. With the optimal choice of coefficients, we would therefore expect to be able to achieve two additional orders of accuracy compared to Forward Euler, i.e., to achieve third order.

b. We enforce that the scheme, when using $h = 1$, is exact for the test functions $1, x, x^2, x^3$, giving the relations

$$\begin{cases} a_1 + a_2 = 1 \\ a_2 - b_0 - b_1 = -1 \\ a_2 + 2b_0 = 1 \\ a_2 - 3b_0 = -1 \end{cases}$$

Starting from the bottom two equations, the system is readily solved, giving

$$\{a_1 = \tfrac{4}{5},\ a_2 = \tfrac{1}{5},\ b_0 = \tfrac{2}{5},\ b_1 = \tfrac{4}{5}\}.$$

c. To test the root condition, we apply the scheme to the ODE $y' = 0$. The resulting three-step linear recursion relation has the characteristic equation $r^2 - \frac{4}{5}r - \frac{1}{5} = 0$, with the two roots $r_1 = 1$, $r_2 = -\frac{1}{5}$. Since the second root is inside the unit circle, the root condition holds.

d. Applying the scheme to the special ODE $y' = \lambda y$, and setting $h\lambda = \xi$ gives the characteristic equation

$$r^2(1 - \tfrac{2}{5}\xi) - r(\tfrac{4}{5} + \tfrac{4}{5}\xi) - \tfrac{1}{5} = 0.$$

We can now plot the stability domain boundary by solving for ξ, setting $r = e^{i\theta}$, and letting θ run over $0 \le \theta \le 2\pi$. (Carrying out these steps analytically and simplifying produces (3).)

e. We see immediately from (3) that the real part always is less than zero when $\theta \neq 0$. The boundary curve (apart from at the origin) is located in the left half plane, which is impossible for an A-stable scheme (for which the whole left half-plane should be within the stability domain).

Alternatively, we can refer to the theorem that an A-stable linear multistep method can at most be of second order of accuracy. Since the present scheme is third order accurate, it cannot be A-stable.

6. **Numerical PDEs:** **Solution:**

a. The difference approximation is $\dfrac{u(x,t+k)-u(x,t)}{k} = \dfrac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}$.

b. Substitute $u(x,t)=\xi^{t/k}e^{i\omega x}$ into the difference approximation above to obtain $\xi = 1+\frac{k}{h^2}2(\cos\omega h-1)$. When ωh varies over $[-\pi,\pi]$, the expression $2(\cos\omega h-1)$ varies over [-4,0], and $\xi = 1+\frac{k}{h^2}2(\cos\omega h-1)$ over $[1-4\frac{k}{h^2},1]$. The latter interval must fit inside [-1,1], implying $\frac{k}{h^2}\le\frac{1}{2}$.

c. The described discretization is the Method of Lines (MOL)- produced ODE system

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \\ \vdots \end{bmatrix} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 1 & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \\ \vdots \end{bmatrix}.$$

By noting that the matrix is symmetric, and using Gersgorin's theorem, we obtain that its eigenvalues (including the factor $1/h^2$) satisfy $\lambda\in\frac{1}{h^2}[-4,0]$. These have all to fit inside the Forward Euler stability domain (which is the inside of a unit radius circle centered at -1), i.e. we need (for all the λ's above) that $\lambda k\in[-2,0]$. This becomes assured if $\frac{k}{h^2}\le\frac{1}{2}$.