Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Monday August 20, 2012 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Nonlinear equations:

1. Suppose that $g : [a,b] \to [a,b]$ is continuous on the real interval $[a,b]$ and is a *contraction* in the sense that there exists a constant $\lambda \in (0,1)$ such that

$$|g(x) - g(y)| \le \lambda\, |x - y| \qquad \text{for all } x, y \in [a,b].$$

Prove that there exists a unique fixed point in $[a,b]$ and that the fixed point iteration $x_{n+1} = g(x_n)$ converges to it for any $x_0 \in [a,b]$. Also, prove that the error is reduced by a factor of at least λ *from each iteration to the next.*

Numerical quadrature:

2. We consider here three different strategies for determining weights in 3-node quadrature approximations of the form

$$\int_0^1 u(x)\,dx \approx \alpha\, u(0) + \beta\, u(\tfrac{1}{2}) + \gamma\, u(1).$$

Determine the quadrature weights (α, β, γ) that are obtained in the following three cases:

a. Trapezoidal rule,
b. Simpson's formula,
c. Exact integration of the interpolating *natural* cubic spline (i.e., the cubic spline across $[0,1]$ with end conditions that the second derivative is zero).

Interpolation / Approximation:

3. Let $f : [a,b] \to \mathfrak{R}$ be a real-valued continuous function on the closed interval $[a,b]$. Suppose that p_n^* solves the minimax problem in the sense that it is a polynomial of degree less than or equal to $n \ge 1$ that minimizes $\max_{x\in[a,b]} |e(x)|$ over all polynomials of degree equal to n, where $e(x) = f(x) - p_n(x)$. Prove that there must exist at least two points $\alpha, \beta \in [a,b]$, such that $|e(\alpha)| = |e(\beta)| = \max_{x\in[a,b]} |f(x) - p_n^*(x)|$ and $e(\alpha) = -e(\beta)$.

Linear algebra:

4. Let $\|\cdot\|$ here denote the Euclidean norm and suppose that $A \in \Re^{n \times n}$ (i.e., A is a real-valued $n \times n$ matrix).

   a. Prove that $\|QAR\| = \|A\|$ when Q and R are $n \times n$ unitary matrices.

   b. Define the singular value decomposition of A.

   c. Prove that $\|A\| = \|A^T\|$.

   d. Prove that the spectral radius of A, denoted here by $\rho(A)$, is bounded by $\|A\|$.

   e. Prove that $\rho(A) = \|A\|$ when A is symmetric.

   f. Illustrate by a simple example that $\|A\|$ can be very much larger than $\rho(A)$.

Numerical ODE:

5. Consider a linear multistep scheme of the form

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + h\,(b_0 f(x_{n+1}, y_{n+1}) + b_1 f(x_n, y_n))$$

for solving the ODE $y' = f(x, y)$.

   a. Based on some general 'rule of thumb', explain what is the highest order of accuracy this scheme can attain.

   b. Determine the coefficients a_1, a_2, b_0, b_1 that makes it reach this order of accuracy.

   c. Determine whether or not the obtained scheme satisfies the root condition.

   d. Write down an equation that describes the edge of the scheme's stability domain.

   e. It transpires that the domain boundary obtained in part (d) above can be expressed explicitly in the form

$$\xi = -\frac{4\sin^4(\theta/2)}{5 - 4\cos(\theta)} + i\left(\frac{(8 + \cos(\theta))\sin(\theta)}{5 - 4\cos(\theta)}\right), \quad 0 \le \theta \le 2\pi.$$

   Determine, based on this (or by some other means), whether the scheme is A-stable.

Numerical PDE:

6. Consider the PDE $u_t = u_{xx}$ and approximate it with Forward Euler in time, centered second order finite differences in space. You can assume that the spatial domain is either periodic or $[-\infty, \infty]$.

   a. Write down the difference equation for this scheme.

   b. Use von Neumann analysis to obtain the stability condition that relates the allowable time step k and space step h.

   c. Obtain the same result via an ODE stability domain-based argument.