1. Root Finding. (a) What is the iteration formula for Newton’s method applied to minimizing a general functional, $\psi(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, where $n$ is a positive integer?
(b) State general assumptions on $\psi(x)$ and a minimizer, $x = \alpha$, that enable a known fixed-point result to establish local convergence of Newton’s method about $\alpha$.
(c) How well does Newton’s method work for $\psi(x) = (1/2) \langle x, Ax \rangle - \langle x, b \rangle$, where $A \in \mathbb{R}^{nxn}$ and $b \in \mathbb{R}^n$ are given? Support your claim by computing the gradient and Hessian of $\psi(x)$ and confirming the method’s behavior.

2. Numerical quadrature. Consider a subspace $P_{N-1}$ of polynomials of degree $N-1$ on the interval $[-1, 1]$ and associated inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.
(a) Using the minimal number of nodes, construct a discrete inner product identical to (1) on $P_{N-1}$. Prove that the two inner products coincide on $P_{N-1}$.
(b) Construct an orthonormal basis in $P_{N-1}$ such that the coefficients of expansion of polynomials in $P_{N-1}$ in this basis are given by the scaled values of these polynomials at the nodes. (Hint: In your construction, use the discrete inner product and the Lagrange interpolating polynomials that are one at one node point and zero at all others.)

3. Interpolation/Approximation. Let $U$ denote the Discrete Fourier Transformation (DFT) matrix
$$U = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\
1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\
1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)^2}
\end{pmatrix},$$
where $\omega$ is the $N^{th}$ root of unity: $\omega = e^{2\pi i/N}$.
(a) Show that $U^*U = N \cdot I$, where $I$ is the $N \times N$ identity matrix.
(b) Let $C$ be a circulant $N \times N$ matrix (a special case of a Toeplitz matrix, such that its entries $c_{i,j}$ depend only on mod($i - j, N$), that is, on the remainder when $i - j$ is divided by $N$). For example, a $4 \times 4$ circulant matrix would be of the form
$$C = \begin{pmatrix}
c_0 & c_1 & c_2 & c_3 \\
c_3 & c_0 & c_1 & c_2 \\
c_2 & c_3 & c_0 & c_1 \\
c_1 & c_2 & c_3 & c_0
\end{pmatrix}.$$ 
Show that $U^*CU = D$, where $D$ is a diagonal matrix containing the DFT of $C$’s first row down its diagonal. (Hint: It suffices to demonstrate this in a $4 \times 4$ case if your derivation makes it very clear how the result immediately generalizes to an $N \times N$ case. One possible approach starts by first considering suitable special cases for the constants $c_k, k = 0, 1, 2, 3$.)
(c) Suppose we want to carry out a matrix $\times$ vector multiplication $Cx = y$, where $C$ (circulant) and $x$ (column vector) are given. Explain how to get $y$ in just $O(N \log N)$ operations.
4. **Linear Algebra.** Consider the matrix
\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \]

(a) Compute the eigenvalues of \( A \) and their associated eigenvectors.
(b) What are the characteristic and minimal polynomials of \( A \)? (Leave them in factored form.)
(c) What are \( A \)'s singular values?

5. **Numerical ODEs.** Consider
\[ y' = f(t, y) \]
with the initial condition \( y(0) = y_0 \) and a Runge-Kutta scheme,
\[ y_{n+1} = y_n + h \left( f(t_n, y_n) + 3f(t_n + \frac{2}{3}h, y_n + \frac{2}{3}hf(t_n, y_n)) \right). \]

a) Find the order of this scheme, define its region of absolute stability, and find a part of the real line that belong to that region.
(b) How would you argue that the method is convergent?

6. **Numerical PDEs.** Consider the PDE \( u_t + iu_{xx} = 0 \) (where \( i = \sqrt{-1} \)) on the domain \(-\infty < x < \infty, t \geq 0\).

(a) We wish to approximate it with standard second-order centered finite differences in space and, for the time integration, use either of the following three options:
   (i) Leap-frog.
   (ii) Forward Euler.
   (iii) Crank-Nicolson.

Tell for each of these three cases the stability condition that the time and space steps will need to satisfy. In one of the cases (does not matter which), demonstrate your result by von Neumann analysis.

(b) Suppose that we decide to approximate the space derivative with a one-sided second-order finite difference approximation, combined with leap-frog in time. Should the one sided stencil extend to the left or to the right of the \( x \)-location at which we advance in time, or does that not matter? What stability condition would we obtain? (Hint: Note that the PDE is satisfied by \( u(x, t) = e^{i\omega^2 t + i\omega x} \).)