Please submit solutions to one of the problems 1, 2 and to 3 of the problems 3 - 6 (and to no more). The test will last 3 hours.

1. **Gaussian quadrature:** This technique is commonly used for rapid evaluation of integrals. One useful generalization is to instead apply it to evaluate infinite (or finite) sums:

   \[ \sum_{n=0}^{\infty} \frac{f(n)}{n!} = w_1 f(x_1) + w_2 f(x_2) \]

   becomes exact for polynomials \( f(x) \) of as high degree as possible.

   **Hint:** Sums of the form \( \sum_{n=0}^{\infty} \frac{n^p}{n!} \) can be found by considering derivatives of \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) at \( x = 1 \).

2. **Comparison of quadrature methods:** Suppose we know the first three derivatives of a function \( y = f(x) \) at both \( x = 0 \) and \( x = 1 \), and that we want to evaluate \( \int_0^1 f(x) \, dx \) as accurately as possible. The following are four plausible approaches. Compare their orders of accuracy, i.e. how high degree polynomials they are exact for (if two methods have the same order, compare the size of their leading error coefficients). Also note if any of the methods feature any particular practical strengths or weaknesses:

   a. Consider the Taylor expansions around \( x = 0 \) and \( x = 1 \), integrate both over the interval \([0,1]\), and use the average of these two results,

   b. Integrate the Taylor expansions over \([0,1/2]\) and \([1/2,1]\) resp., and add the results,

   c. Form the (generalized) Hermite interpolation polynomial that uses function value and three derivatives at each end, and integrate this polynomial exactly,

   d. Use the first three terms in Euler-MacLaurin's formula.

   **Note:** The resulting formulas can be written as follows:

   a. \( \int_0^1 f(x) \, dx = \frac{1}{2}[f(0) + f(1)] + \frac{1}{4}[f'(0) - f'(1)] + \frac{1}{12}[f''(0) + f''(1)] + \frac{1}{48}[f'''(0) - f'''(1)] \),

   b. \( \int_0^1 f(x) \, dx = \frac{1}{2}[f(0) + f(1)] + \frac{3}{8}[f'(0) - f'(1)] + \frac{5}{48}[f''(0) + f''(1)] + \frac{1}{384}[f'''(0) - f'''(1)] \),

   c. \( \int_0^1 f(x) \, dx = \frac{1}{2}[f(0) + f(1)] + \frac{1}{8}[f'(0) - f'(1)] + \frac{7}{48}[f''(0) + f''(1)] + \frac{1}{16632}[f'''(0) - f'''(1)] \),

   d. \( \int_0^1 f(x) \, dx = \frac{1}{2}[f(0) + f(1)] + \frac{5}{12}[f'(0) - f'(1)] - \frac{1}{720}[f'''(0) - f'''(1)] \).

   The orders can be established by 'brute force' testing with \( f(x) = 1, x, x^2, x^3, \ldots \). DO NOT solve the problem this way, but use instead general arguments based on how the methods were designed.
3. **Iteration methods for root finding:** We want to converge to the root \( x = 0 \) of \( f(x) = x - \sin x \) by iteration.
   a. If we start with \( x_0 = 1 \), estimate roughly how many iterations it will take to reach an error of \( 10^{-6} \) when using
      i. \( x_{n+1} = \sin x_n \),
      ii. \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \),
   b. Tell how the scheme in part a.ii can be altered to achieve quadratic convergence. Give a general justification showing that your suggestion will indeed work.

4. **Stability test for finite difference approximations to a PDE:** We want to numerically solve the initial value problem for the PDE \( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \).
   a. State which of the following schemes i - v (as illustrated by their stencils below) that are
      - unconditionally unstable,
      - conditionally stable, (state the stability condition)
      - unconditionally unstable?
      i. leap-frog
      ii. forward
      iii. downwind
      iv. upwind
      v. Crank-Euler Nicolson
   b. The CFL (Courant-Friedrichs-Lewy)-condition states that a scheme must be unstable if the stencil geometry is such that it cannot propagate information in a direction of a characteristic. Find the ratios of time- to space step that this condition rules out (for possible stability) in the five cases above.
   c. In one of the cases above, apply the von Neuman stability test (called 'Fourier series method' in Hall & Porsching), to strictly verify your assertions under part a.

5. **Linear multistep methods for ODEs:** The two simplest (and least efficient!) of all schemes to solve \( y' = f(x, y) \) are probably the Forward- and Backward Euler schemes (also known as the Adams-Bashforth- and Adams-Moulton methods of order one):
   a. State the general root condition for convergence, and apply it to the two schemes,
   b. Describe what is meant by 'stability domain', and establish these for the two schemes.
6. **Numerical linear algebra:** Householder matrices $H = I - 2\omega\omega^*$ (with $\omega^*\omega = 1$) form one of the most important 'building blocks' in numerous key algorithms.

   a. Show that $H$ is both Hermitian and unitary.
   
   b. Given two vectors $x$ and $y$, describe when one can find an $H$ such that $Hx = y$.
   
   c. Assume the condition(s) in part b above is (are) met, describe, how one then finds the appropriate vector $\omega$ (and thus matrix $H$)

   You need only to state how one chooses the vector $\omega$, and give some sort of justification for why this choice is the appropriate one. You do NOT need to work through the algebra to formally verify that your choice indeed achieves $Hx = y$ (that is somewhat tricky).

   d. Describe how similarity transformations with Householder matrices can be employed to bring a general square matrix $A$ into upper Hessenberg form.

   e. Assuming $A$ is of size $n \times n$, count (to leading order) the number of arithmetic operations that is required by the similarity transform in part c.