Applied Analysis Preliminary Exam (Hints/solutions)
1:30 PM – 4:30 PM, January 9, 2020

Instructions You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

1. The following two problems are unrelated.

   (a) Let X and Y be normed vector spaces, and $D \subset X$ a convex subset of X. If $f : D \to Y$ is Hölder continuous with exponent $\alpha > 1$, prove that in fact f is a constant function.
   Hint: recall Hölder continuity with exponent α means $\|f(x) - f(x')\|_Y \leq c \cdot \|x - x'\|_X^\alpha$ for some constant c, for all $x, x' \in D$.
   Hint: you may wish to apply the triangle inequality repeatedly.

   Solution: Fix any $x, x' \in D$ and partition the line segment $\overline{xx'}$ into n equispaced segments with end points $\{x = x_0, x_1, x_2, \ldots, x_n = x'\}$. By convexity of D, all these points are also inside D. The length of any segment $\overline{x_i x_{i-1}}$ is $\|x_i - x_{i-1}\| = \|x - x'\|/n$. Then by the triangle inequality,

$$\begin{aligned}\|f(x) - f(x')\| = \left\|\sum_{i=1}^n f(x_i) - f(x_{i-1})\right\| &\leq \sum_{i=1}^n \|f(x_i) - f(x_{i-1})\| \\ &\leq \sum_{i=1}^n c\|x_i - x_{i-1}\|^\alpha \\ &= \sum_{i=1}^n c\left(\frac{\|x - x'\|}{n}\right)^\alpha \\ &= cn\left(\frac{\|x - x'\|}{n}\right)^\alpha \\ &= cn^{1-\alpha}\|x - x'\|^\alpha\end{aligned}$$

   which can be made arbitrarily small by choosing n sufficiently large (since $1 - \alpha < 0$), hence $f(x) = f(x')$ for all $x, x' \in D$, so f is a constant function.

   (b) Consider the following variant of the Weierstrass Function: $f(x) = 2\sum_{n=1}^\infty a^n \cos(b^n x)$ with $a = \frac{1}{8}$ and $b = 49$ (this function is not differentiable at any point). Prove

   i. that $f \in L^2(\mathbb{T})$ where $\mathbb{T}$ is the 1D torus of length 2π, and
   ii. that f is continuous.

   Solution: We can expand f in Fourier series as $f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$ where

$$c_k = \begin{cases} a^n & k = b^n \\ a^n & k = -b^n \\ 0 & \text{otherwise.}\end{cases}$$

   Note $c_k e^{ikx} + c_{-k}e^{-ikx} = 2\cos(kx)$, so the Fourier series reduces to the cosine series given in the problem statement. To prove $f \in L^2(\mathbb{T})$, we only need to show $\sum_{k\in\mathbb{Z}} c_k^2 < \infty$, which is the same as showing $2\sum_{n=1}^\infty (a^2)^n < \infty$. This is true since $a < 1$ so $a^2 < 1$, so this is a convergent geometric series. To prove f is continuous, we'll use the Sobolev embedding theorem, and prove $f \in H^s(\mathbb{T})$ for $s > 1/2$. Recall $f \in H^s(\mathbb{T})$ iff $\sum_{k\in\mathbb{Z}} |k|^{2s} c_k^2 < \infty$. Removing the zero coefficients from this series, we can rewrite it as $2\sum_{n=1}^\infty |b^n|^{2s}(a^n)^2$, i.e.,

$2\sum_{n=1}^{\infty}\left((b^s a)^2\right)^n$, and so this is a convergent geometric series if $b^s a < 1$. For $s = 1/2$, we have $b^s a = \sqrt{49}/8 = 7/8 < 1$. Since $s \mapsto b^s a$ is a continuous function, it follows that there is some value of $s > 1/2$ for which $b^s a < 1$, and hence we can use the Sobolev embedding theorem to conclude f is continuous.

Alternative solution: First, prove (ii) (that f is continuous). This follows from the Weierstrass M-test, since we can write f as the uniform limit of continuous functions. Then (i) (that $f \in L^2(\mathbb{T})$) follows easily because it's clear $C([0, 2\pi]) \subset L^2(\mathbb{T})$.

2. Let $I = [0, 1]$ and $k : I^2 \to \mathbb{R}$ be a continuous function. Fix some $1 \le p \le \infty$, and define

$$\forall f \in L^p(I),\ \forall x \in I, \quad \big(Tf\big)(x) = \int_0^1 k(x, y) f(y)\, dy.$$

(a) Prove that Tf is a continuous function on I.

Solution: Fix some $x \in I$ and consider some $x' \in I$ with $|x' - x| < \delta$. Then

$$\begin{aligned}
\big|\big(Tf\big)(x) - \big(Tf\big)(x')\big| &= \left|\int_0^1 k(x, y) f(y) - k(x', y) f(y)\, dy\right| \\
&\le \int_0^1 |k(x, y) - k(x', y)| \cdot |f(y)|\, dy \\
&\le \sup_{y \in I} |k(x, y) - k(x', y)| \cdot \|f\|_1 \quad \text{via Hölder's ineq.}
\end{aligned}$$

and since k is jointly continuous in (x, y), by choosing δ sufficiently small, we can make the above bound arbitrarily small, thus showing Tf is continuous.

Note: we use the fact that $\|f\|_1 < \infty$. This follows from $f \in L^p(I)$ since I is bounded (it is not true that $L^p \subset L^1$ on unbounded domains). To prove that $f \in L^1(I)$, there are several ways. Here's one method: divide I into two parts, $I_1 = \{x \mid |f(x)| \le 1\}$ and $I_2 = \{x \mid |f(x)| > 1\}$ so $I = I_1 \cup I_2$. Then

$$\|f\|_1 = \int_I |f(x)| = \int_{I_1} |f(x)|\, dx + \int_{I_2} |f(x)|\, dx \le \int_{I_1} 1\, dx + \int_{I_2} |f(x)|^p\, dx \le 1 + \|f\|_p^p$$

since $|f(x)| \le |f(x)|^p$ if $|f(x)| \ge 1$ and $p \ge 1$.

(b) Prove the image of the unit ball in $L^p(I)$ is pre-compact in $C(I)$

Solution: The proof is basically just invoking the Arzela-Ascoli theorem, which says that if I is compact, then subsets of $C(I)$ are precompact if they are bounded and equicontinuous (and compact if they are additionally closed).

If f is in the unit ball of $L^p(I)$, then by our calculation above, $\|f\|_1 \le 2$, and hence, also by similar inequalities to above, $\|Tf\|_\infty \le \|k\|_\infty \|f\|_1 \le 2\|k\|_\infty$, so we see that the image is bounded.

To show equicontinuity, we need that the δ in the continuity definition does not depend on which f we have. But this also follows from our calculations above, since we can bound $\|f\|_1 \le 2$, so the value of δ depends only on k (and the target accuracy ϵ), not f, hence the set $\{Tf \mid f \in L^p(I),\ \|f\|_p \le 1\} \subset C(I)$ is bounded and equicontinuous, hence (via Arzela-Ascoli) it is precompact.

3. Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space.

(a) Let $\lambda \neq 0$ be in the point spectrum of A, and define the corresponding eigenspace M_λ to be the set of all associated eigenvectors. Prove M_λ is a Hilbert space.

Solution: Since M_λ is the kernel of $(A - \lambda I)$ (where I is the identity), and $(A - \lambda I)$ is a continuous operator (since it is linear and bounded), it follows M_λ is closed using sequential continuity. It's also a subspace since the null space is always a subspace (again, easy to show; use linearity). Hence it's a closed subspace of a Hilbert space, so it itself is a Hilbert space.

(b) If we also assume that A is a compact operator, prove that M_λ must be finite dimensional

Solution: Note that M_λ is a Hilbert space, so it has some orthonormal basis, and "dimension" means the size of this orthonormal basis. For contradiction, suppose it is not finite dimensional, thus the orthonormal basis is infinite, and one can take a sequence (e_n) of orthonormal basis elements. This is a bounded sequence (since e_n is normalized), so by compactness of A, it means (Ae_n) has a convergent subsequence. But $Ae_n = \lambda e_n$ since e_n is an eigenvector. The sequence (λe_n) is not Cauchy since it is orthogonal ($\|\lambda e_n - \lambda e_m\|^2 = \lambda^2 2$ for $n \neq m$, and recall $\lambda \neq 0$), and similarly it follows that all subsequences are also not Cauchy, hence there cannot be any convergent subsequence, which is a contradiction.

4. Let X be a normed linear space. If $x_n \rightharpoonup x$ in X, prove $\|x\| \leq \liminf \|x_n\|$.

Solution: This is a generalization of Prop. 8.44 in the book. Via the Hahn-Banach theorem, for any fixed $x \neq 0$, there exists $\varphi \in X^*$ such that $\varphi(x) = \|x\|$ and $\|\varphi\| = 1$ (this is exercise 5.6 in the book; you can state it without proof). Thus

$$\begin{aligned} \|x\| = \varphi(x) &= \lim \varphi(x_n) \\ &= \liminf \varphi(x_n) \\ &\leq \liminf \|\varphi\| \cdot \|x_n\| \\ &= \liminf \|x_n\|. \end{aligned}$$

(We changed the lim to lim inf since after doing the bound, we're not sure if the new sequence converges or not). If $x = 0$, the above proof doesn't work, but then $\|x\| = 0$ and the statement is trivial.

5. If $f \geq 0$ is measurable and $\int_E f \, d\mu = 0$ (where μ is the Lebesgue measure), prove that $f(x) = 0$ almost everywhere on E.
Hint: define $E_n = \{x \mid f(x) > 1/n\}$ and consider $\bigcup_{n=1}^\infty E_n$. Note that measures are countably sub-additive.

Solution: Define E_n as in the hint, and note that this is a measurable set. Then since $f \geq 0$, it follows $0 = \int_E f \, d\mu \geq \int_{E_n} f \, d\mu \geq \frac{1}{n}\mu(E_n)$ which implies $\mu(E_n) \leq 0$ (and hence $\mu(E_n) = 0$ by non-negativity of measure) for all n. By countable sub-additivity,

$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n) = 0$$

meaning that the set of x where $f(x) \neq 0$ has zero measure, i.e., $f(x) = 0$ almost everywhere.