Applied Analysis Preliminary Exam (Hints/solutions)
10:00 AM – 1:00 PM, January 17, 2018

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

1. The following two questions are unrelated.

   (a) Prove the following series converges:

$$\sum_{k=1}^{\infty} k^3 e^{-k}$$

   Solution: One idea: Taylor expand e^x (for $x > 0$) and use the remainder theorem to see it dominates any polynomial, so bound this with a polynomial (i.e., the "comparison test"); for example, use $e^k > k^5/(5!)$, so $\sum_{k=1}^{\infty} k^3 e^{-k} \le 1/(5!) \sum_{k=1}^{\infty} k^{-2}$. Then use the "p-test" to see that sum $1/k^p$ converges iff $p > 1$. The series is non-negative, so it is bounded above, and hence the monotone convergence theorem (for series) implies that it converges.

   Another idea: use the integral test (evaluate integrals via integration-by-parts). By taking derivatives, you can see that $f(x) = x^3 e^{-x}$ is monotonically decreasing for $x > 3$.

   (b) Let $\{f_n \in C([0,1]) \mid n \in \mathbb{N}\}$ be equicontinuous. If $f_n \to f$ pointwise, prove that f is continuous.

   Solution: Note that this is 2.12 from the book, and was problem 8 on homework 5 (2017).

   Fix some $\epsilon > 0$ and any $x \in [0,1]$, and we will show there is some $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. By equicontinuity, there is some $\delta > 0$ such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon/3$ for all n. By point wise convergence, there is some N_x such that $|f_n(x) - f(x)| < \epsilon/3$ for all $n \ge N_x$, and there is some N_y such that $|f_n(y) - f(y)| < \epsilon/3$ for all $n \ge N_y$. Now, pick any y with $|x - y| < \delta$, and then choose any $n \ge \max N_x, N_y$, we have

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

   For full-credit, the student needs to be clear where they are using equicontinuity and not just continuity, since the result is not true without assuming equicontinuity.

   Another approach: also using compactness of $[0,1]$ and (uniform) equicontinuity of (f_n), we can either show it directly (as above) or use the triangle inequality to show that (f_n) is bounded. It is clear that $(f_n(x))$ is bounded for all x, but we need to use the continuity and compactness to show that this bound doesn't grow arbitrarily large for different x. Once we have it bounded, then we can apply Arzela-Ascoli to get sequential compactness and conclude that some subsequence, (f_{n_k}), converges uniformly to some continuous function g. But then for all x, $f_{n_k}(x)$ also converges to $f(x)$, so $f(x) = g(x)$ for all x, hence $f = g$ so we conclude f is continuous.

2. Prove that if $0 \le \lambda \le 1$, then the equation

$$u(x) = \lambda \int_0^1 \frac{1}{1 + x + u(s)}\, ds, \quad \text{for} \quad 0 \le x \le 1$$

has exactly one continuous non-negative solution. *Hint: if u is a solution, what can you say about its maximum value? and about its minimum value?*

Solution: Work with $C([0,1])$ and the uniform norm. We can make the subset $C_+([0,1]) \subset C([0,1])$ consisting of non-negative continuous functions, and since uniform convergence implies pointwise convergence, it's easy to see that C_+ is a closed (hence complete) space. Define $T : C_+([0,1]) \to C_+([0,1])$ by $(Tu)(x) = \lambda \int_0^1 \frac{1}{1+x+u(s)}\, ds$. We can verify that indeed the range is inside $C([a,b])$ by the fundamental theorem of calculus, and if u is non-negative, the integrand is non-negative, so Tu is non-negative, so the range is inside $C_+([0,1])$.

In fact, we are going to "boot-strap" a bit to find some better bounds on Tu. We saw that if $u \geq 0$ then $Tu \geq 0$, and from this, we conclude an upper bound for $x \in [0,1]$,

$$(Tu)(x) \leq \frac{\lambda}{1+x}.$$

Suppose we further restrict our domain to functions $u(x) \leq \lambda/(1+x)$ (again, this set is closed).[Note: if you further bound this by just $u(x) \leq \lambda$, it works too.] Now, here is the "boot-strapping" part: we can a new better lower bound on Tu:

$$(Tu)(x) \geq \lambda \int_0^1 \frac{1}{1+x+\frac{\lambda}{1+x}}\, ds = \frac{\lambda(1+x)}{(1+x)^2+\lambda}.$$

We can even further restrict our domain to include this new lower bound as well (call this domain just "C"), and so we know $T : C \to C$. To make the new lower bound simpler, using $x \geq 0$ in the denominator and $x \leq 1$ in the numerator, we have $(Tu)(x) \geq \lambda/(4+\lambda)$. [There are other variants possible: if we earlier showed $u(x) \leq \lambda$ and plugged that in, then we have $u(x) \geq \lambda/(2+\lambda)$ using $x \leq 1$.]

[Be careful with the following reasoning: noting that $u \equiv 0$ is not a solution, and that u is continuous, we can require that our set C can be restricted to continuous functions that are nonzero, and hence there is some set of non-zero measure where $u > \delta > 0$. But the problem with this is that this restriction of C is no longer a closed subset of $C([0,1])$, hence not complete].

Now we wish to show T is contractive in the uniform norm, so the statement of the problem then follows from the Banach fixed point (aka contraction mapping) theorem.

$$\begin{aligned}
\|Tu - Tv\|_u &= \sup_x \lambda \left| \int_0^1 \frac{1}{1+x+u(s)} - \frac{1}{1+x+v(s)}\, ds \right| \\
&= \sup_x \lambda \left| \int_0^1 \frac{(1+x+v(s)) - (1+x+u(s))}{(1+x+u(s))(1+x+v(s))}\, ds \right| \\
&\leq \sup_x \lambda \int_0^1 \frac{|u(s)-v(s)|}{(1+x+u(s))(1+x+v(s))}\, ds \\
&\leq \sup_x \lambda \int_0^1 \frac{|u(s)-v(s)|}{(1+u(s))(1+v(s))}\, ds \\
&\leq \lambda \int_0^1 \frac{|u(s)-v(s)|}{(1+\lambda/(4+\lambda))^2}\, ds \\
&\leq \|u-v\|_\infty \frac{\lambda}{(1+\lambda/(4+\lambda))^2}
\end{aligned}$$

and $\lambda \mapsto \frac{\lambda}{(1+\lambda/(4+\lambda))^2}$ is continuous and increasing and bounded strictly by 1 on $[0,1]$ (in fact, we can go until about $\lambda \lesssim 1.66$). [If we bounded $u(x) \geq \lambda/(2+\lambda)$ earlier, then we can bound the numerator by $1 + 2\lambda/(2+\lambda)$, and get that we have a contraction if $\lambda^2 - \lambda - 2 < 0$, i.e., if $\lambda \in (-1,2)$]. Hence T is a contraction on C if $0 \leq \lambda \leq 1$.

3. The following two questions are unrelated.

   (a) Let X and Y be Banach spaces and let $T : X \to Y$ be a continuous linear bijection. Prove that there exists a positive constant β such that

   $$\|Tx\| \geq \beta \|x\| \quad \text{for all } x \in X.$$

   Solution: Follows immediately from the open mapping theorem, which proves T^{-1} is continuous and hence bounded (since it is linear). It is also possible to cite Prop. 5.30 from Hunter and Nachtergaele's book.

   (b) i. Consider the bounded linear operator $T : \ell^2 \to \ell^2$ defined by, for $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$,

   $$(Tx)_n = \frac{1}{n} x_n.$$

   Prove that 0 is in the spectrum of T.

   Solution: For grading, we want to make sure the student knows the different types of spectrum (point spectrum/eigenvalues, residual spectrum, continuous spectrum). The value $\lambda = 0$ is not an eigenvalue, but it is in the spectrum because $T - \lambda I$ (for $\lambda = 0$) is not onto.

   There are several ways to show that it is not onto. One way: note that if it were onto, then we satisfy the assumptions of the open mapping theorem, so T^{-1} would be a bounded linear operator, but it's clear that T^{-1} cannot be bounded. Another way: note that $\lambda_n = 1/n$ is clearly an eigenvalue, and that the spectrum is closed (i.e., the resolvent set is open), so therefore the limit point $\lambda = 0$ must be in the spectrum.

   ii. For the same linear operator T from part (i), prove or disprove that $I + T$ is a compact operator (I is the identity).

   Solution: While T is compact, I is not compact, and $I + T$ is not compact either. To show this, consider the bounded sequence of canonical basis vectors e_n. Then $(I + T)(e_n) = (1 + 1/n)e_n$ and this sequence is not Cauchy, hence it has no convergent subsequences, so $(I + T)$ does not map bounded sets to precompact sets, so it is not a compact operator.

   Note: for full credit, you need some argument. It is not OK to just say that since I is not compact, therefore $I + T$ is not compact. For example, if we took $T = -I$, then $I + T$ is compact.

4. Let f belong to the Sobolev space $H^1(\mathbb{T})$. Prove there exists a unique function $g \in L^2(\mathbb{T})$ such that

   $$\int_{\mathbb{T}} g\varphi \, dx = -\int_{\mathbb{T}} f\varphi' \, dx \quad \forall \varphi \in C^1(\mathbb{T}).$$

   Solution:

   Let $(\widehat{f}_n)$ and $(\widehat{\varphi}_n)$ be the Fourier coefficients of f and φ, respectively. Then $((in)\widehat{\varphi}_n)$ are the Fourier coefficients of φ'. Via Parseval's theorem,

   $$\begin{aligned} T(\varphi) := -\int_{\mathbb{T}} f\varphi' \, dx &= -\langle f, \varphi' \rangle_{L^2} \\ &= -\sum_{n=-\infty}^{\infty} \overline{\widehat{f}_n} \cdot (in\widehat{\varphi}_n) \\ &= -\sum_{n=-\infty}^{\infty} \overline{-in\widehat{f}_n} \cdot (\widehat{\varphi}_n) \\ &\leq \|f\|_{H^1} \cdot \|\varphi\|_2 \end{aligned}$$

   showing that T is a bounded linear functional on C^1 (it is also OK if the students cite Definition 7.7 in Hunter and Nachtergaele to jump to this fact). It acts on C^1, which is dense in L^2, so by the BLT theorem, it can be extended uniquely to a bounded linear functional $\overline{T}$ on L^2. By the Riesz representation theorem, there is an element $g \in L^2$ such that $\overline{T}(\varphi) = \langle g, \varphi \rangle$, which gives the desired equality. This element g is called the "weak derivative" of f, and is usually denoted by f'.

5. Let $(q_i)_{i \in \mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and let λ denote the Lebesgue measure. Consider the functions f_n, for $n \in \mathbb{N}$, defined on $[0,1]$ as

$$f_n(x) = \begin{cases} 1 & \text{if } x = q_i \text{ for some } i \leq n \\ 0 & \text{else.} \end{cases}$$

   (a) Prove f_n is a Lebesgue measurable function

   (b) Let $f(x) = \lim_{n \to \infty} f_n(x)$ for $x \in [0,1]$. Is $\int_0^1 f \, d\lambda$ defined? If so, calculate or bound the value of the integral, if possible. Justify your work.

   (c) Does the Riemann integral of f_n exist? Does the Riemann integral of f exist? Very briefly justify your work.

Solution:

   (a) The quickest proof is observing that f_n is a simple function. Otherwise, note that f_n is Lebesgue measurable iff for every $c \in \mathbb{R}$, the set $A_c = \{x \mid f_n(x) > c\}$ is a Lebesgue measurable set (Prop. 12.23, and change $<$ to $>$ for convenience, which follows from properties of a σ-algebra). For $c < 0$, the set $A_c = [0,1]$ which is measurable; for $c \geq 1$, the set $A_c = \emptyset$ which is measurable; and for $0 \leq c < 1$, A_c is a finite collection of points in $[0,1]$, and it is measurable (similiar to Example 12.12, i.e., following arguments of Thm 12.10 about inner/outer approximations by closed/open sets).

   (b) Straightforward from monotone convergence theorem, since $f_n \geq 0$ and are monotone increasing. The value of the integral is zero.

   (c) Each f_n is piecewise continuous with only finitely many discontinuities, so it is Riemann integrable, and the integral is zero. The limiting function f is the Dirichlet function, which is not Riemann integrable. It is clearly nowhere continuous.