Applied Analysis Preliminary Exam
10:00 AM – 1:00 PM, January 17, 2018

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

1. The following two questions are unrelated.
   (a) Prove the following series converges:
   $$\sum_{k=1}^{\infty} k^3 e^{-k}$$
   (b) Let $\{f_n \in C([0,1]) \mid n \in \mathbb{N}\}$ be equicontinuous. If $f_n \to f$ pointwise, prove that f is continuous.
2. Prove that if $0 \le \lambda \le 1$, then the equation
   $$u(x) = \lambda \int_0^1 \frac{1}{1+x+u(s)}\, ds, \quad \text{for} \quad 0 \le x \le 1$$
   has exactly one continuous non-negative solution. *Hint: if u is a solution, what can you say about its maximum value? and about its minimum value?*
3. The following two questions are unrelated.
   (a) Let X and Y be Banach spaces and let $T : X \to Y$ be a continuous linear bijection. Prove that there exists a positive constant β such that
   $$\|Tx\| \ge \beta \|x\| \quad \text{for all } x \in X.$$
   (b) i. Consider the bounded linear operator $T : \ell^2 \to \ell^2$ defined by, for $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$,
   $$(Tx)_n = \frac{1}{n} x_n.$$
   Prove that 0 is in the spectrum of T.
   ii. For the same linear operator T from part (i), prove or disprove that $I + T$ is a compact operator (I is the identity).
4. Let f belong to the Sobolev space $H^1(\mathbb{T})$. Prove there exists a unique function $g \in L^2(\mathbb{T})$ such that
   $$\int_{\mathbb{T}} g\varphi\, dx = -\int_{\mathbb{T}} f\varphi'\, dx \quad \forall \varphi \in C^1(\mathbb{T}).$$
5. Let $(q_i)_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and let λ denote the Lebesgue measure. Consider the functions f_n, for $n \in \mathbb{N}$, defined on $[0,1]$ as
   $$f_n(x) = \begin{cases} 1 & \text{if } x = q_i \text{ for some } i \le n \\ 0 & \text{else.} \end{cases}$$
   (a) Prove f_n is a Lebesgue measurable function
   (b) Let $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0,1]$. Is $\int_0^1 f\, d\lambda$ defined? If so, calculate or bound the value of the integral, if possible. Justify your work.
   (c) Does the Riemann integral of f_n exist? Does the Riemann integral of f exist? Very briefly justify your work.