Instructions:
You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1. (a) Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be the function \( f(x, y, z) = x^2 + 3y \). Find the minimum value of \( f \) on the unit sphere \( S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \).
(b) For what values of \((a, b)\) is the map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( f(x, y) = (ay, y^2 + y^3 - bx) \) a diffeomorphism? When this is the case, find \( f^{-1} \).

2. Let \((T f)(x) = \int_{[0,1]} |x - y|^{1/2} f(y)dy\) for integrable function \( f \).
(a) Show that \( T \) is a compact operator from \( H = L^2(0,1) \) to \( H \).
(b) Show that the equation \( f(x) = g(x) + \int_{[0,1]} |x - y|^{1/2} f(y)dy \)
has a solution \( f \in L^2[0, 1] \) for every \( g \in L^2[0, 1] \).
(c) Show that \( \sigma(T) \subset [-1, \sqrt{3}/3] \) where \( \sigma(T) \) is the spectrum of \( T \).

3. Let \( f_n(x) \) and \( g_n(x) \) be measurable functions on \([0, 1]\), such that \( f_n \to 1/x \) in \( m. \) and \( g_n \to 1/x \) in \( m. \) on \([0, 1]\). Show that \( f_n g_n \to 1/x^2 \) in \( m. \) on \([0, 1]\).

4. Denote by \( Z \) the spaces of all classes of a.e. real-valued measurable functions \( f \) on \([0, 1]\), with two functions \( f \) and \( g \) in the same class iff \( f = g \) a.e. on \([0,1]\). Define on \( Z \) the metric:
\[ \rho(f, g) = \int_{[0,1]} \frac{|f - g|}{1 + |f - g|}. \]
Show that with this metric (you don’t need to show this is a metric) \( Z \) is complete.

5. Prove the existence of a \( C^2 \) solution to the initial value problem
\[ u''(t) + u'^2 + t^2 u + e^u = 0, \quad \text{with} \ u(0) = 0, \ u'(0) = 0. \]
for \( t \in [0, \delta] \), for some \( \delta > 0 \).