Applied Analysis Preliminary Exam, SOLUTIONS

10.00am–1.00pm, August 21, 2018

Problem 1 Solution:

(a) Fix any $x, y \in \mathbb{R}^n$. We want to show that $(g-h) \leq 0$. Note that

$$(g-h)(0) = 0 \text{ and } (g-h)(1) = 0.$$

Suppose that there exists a $t_* \in (0,1)$ for which $(g-h)(t_*) > 0$. By the $\boxed{\text{MVT}}$, we can find a $\xi \in (0, t_*)$ such that

$$(g-h)'(\xi) = \frac{(g-h)(t_*)}{t_*} > 0$$

and an $\eta \in (t_*, 1)$ such that

$$(g-h)'(\eta) = -\frac{(g-h)(t_*)}{1-t_*} < 0$$

Note that $\xi < \eta$ so this contradicts the fact that $(g-h)'$ is monotonically increasing. Therefore, we must have $(g-h) \leq 0$ which implies that f is convex since x, y were arbitrary.

(b) (1) $\boxed{\text{Stokes' theorem}}$ states that if S is an oriented surface in $\mathbb{R}^3$ with piecewise-smooth closed boundary C, and $\mathbf{F}$ is a vector field with each component having continuous partial derivatives on S, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$ where $\mathbf{n}$ is the unit vector normal to S. Since $\nabla \times \mathbf{F} = 0$, this integral is always zero, i.e., the integral along any simple curve is 0. So to integrate from $\mathbf{y}$ to $\mathbf{x}$, choose any curve, and you get the same value (if you go along a different curve back from $\mathbf{x}$ to $\mathbf{y}$ and combine them, then you get 0, so this other curve had the same integral once you account for the change of sign).

(2) The $\boxed{\text{fundamental theorem for gradients}}$ (also known as the fundamental theorem for line integrals) states that if $\nabla\varphi$ is a continuous vector field on an open connected region U in $\mathbb{R}^2$ or $\mathbb{R}^3$, then for all $\mathbf{x}, \mathbf{y} \in U$, $\varphi(\mathbf{x}) - \varphi(\mathbf{y}) = \int_C \nabla\varphi \cdot d\mathbf{r}$ where C is any piecewise-smooth oriented curve in U connecting $\mathbf{x}$ and $\mathbf{y}$. Thus

$$\int_{\mathbf{y}}^{\mathbf{x}} (\nabla\varphi) \cdot d\mathbf{r} = \varphi(\mathbf{x}) - \varphi(\mathbf{y}) = \int_0^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r} - \int_0^{\mathbf{y}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{y}}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r}$$

Problem 2 Solution:

- We need to show that, for all $x \in \mathcal{C}([0,1])$, Fx is also in $\mathcal{C}([0,1])$. It is enough to show continuity of
$$(Ax)(t) := \int_0^1 K(s,t,x(s))\,ds.$$
To this end, consider the set
$$\Omega := \{(s,t,x(s)) : s,t \in [0,1]\}.$$
By continuity of x, Ω is compact in $[0,1]\times[0,1]\times\mathbb{R}$. Since K is continuous, we then have that K is uniformly continuous on Ω.
Let $\varepsilon > 0$. Then $\exists\ \delta > 0$ such that
$$d((s,t_1,x(s)),(s,t_2,x(s))) < \delta \ \Rightarrow\ |K(s,t_1,x(s)) - K(s,t_2,x(s))| < \varepsilon.$$
Thus,
$$|Ax(t_1) - Ax(t_2)| = \cdots < \varepsilon.$$
So, $Ax(t)$ is continuous.
- We next need to show that F is a contraction mapping.
$$\begin{aligned}
|Fx - Fy| &= \max_{0\le t\le 1} |Ax(t) - Ay(t)| \\
&= \max_{0\le t\le 1} \left| \int_0^1 K(s,t,x(s))ds - \int_0^1 K(s,t,y(s))\,ds \right| \\
&\le \max_{0\le t\le 1} \int_{0,1} |K(s,t,x(s)) - K(s,t,y(s))|\,ds \\
&\le \max_{0\le t\le 1} \theta \int_{0,1} |x(s) - y(s)|ds \\
&= \theta \textstyle\int_{0,1} |x(s)-y(s)|ds \le \theta \max\limits_{0\le s\le 1} |x(s) - y(s)|
\end{aligned}$$
Since $0 < \theta < 1$, F is a contraction and we have a unique solution.

Problem 3 Solution:

(a) Let $\mathcal{H} = L^2([0,1])$ and define the operator $T : \mathcal{H} \to \mathcal{H}$ as
$$(Tf)(x) = x \cdot f(x)$$

(1) Determine the point, continuous and residual spectrum of T (with brief justification).

Solution: This is example 9.5 in the Hunter & Nachtergaele book. The answer is that the point and residual spectrum are empty, and the continuous spectrum is the set $[0,1]$. Since T is bounded and self-adjoint, the conclusion that the residual spectrum is empty, and that the rest of the spectrum is real and contained in $[-1,1]$, follows automatically, but for more specifics you just have to calculate (as done in the book's example) — just show that for $\lambda \notin [0,1]$ that $(T - \lambda I)f = g$ has a solution $f \in L^2$.
A common mistake was only doing enough work to prove $[0,1] \subset \sigma_c$ but not really showing $\sigma_c \subset [0,1]$.

(2) Is T compact? Prove your answer.

Solution: One method is to use the spectral theorem. Since T is bounded and self-adjoint, then if it were compact, we could apply the spectral theorem, but since it has no eigenvalues, the spectral theorem would imply that T is the zero operator, which it is not (or similarly, the spectral theorem implies that the continuous spectrum is contained in $\{0\}$, which it isn't). Hence $\boxed{\text{T is not compact}}$.

Another approach is to look at something like (e_n) where (e_n) is an orthonormal basis for $L^2([.5, 1])$ (and define $e_n(x) = 0$ for $x \in [0, .5)$). So it is not an orthonormal basis for $L^2([0, 1])$ but it is still orthonormal. Thus (e_n) is a bounded sequence (it was not important for it to be orthonormal, there are many choices you could make for explicit sequences). An operator T is compact iff for every bounded sequence (e_n), (Te_n) has a convergent subsequence. So does (Te_n) have a convergent subsequence? Since $.5 \le x \le 1$, we can bound $\|Te_n - Te_m\| \ge c > 0$ for some constant, which means (Te_n) cannot be Cauchy, hence cannot have a convergent subsequence. Hence T cannot be compact.

(b) Consider the space $\mathcal{C}([0, 1])$ with the norm $\|f\| = \sup_{x\in[0,1]} |x \cdot f(x)|$. Is this a valid norm? If so, then is this space Banach? Prove your answers.

Solution: This is a valid norm: check that $\|f\| \ge 0$ and $\|f\| = 0$ iff $f = 0$; and check $\|\alpha f\| = |\alpha| \cdot \|f\|$, and check the triangle inequality.

Is the space Banach? No, it cannot be. Let $X = \mathcal{C}([0, 1])$ with the usual uniform norm $\|f\|_u$, and let $Y = \mathcal{C}([0, 1])$ with this new norm. Consider the identity map $I : X \to Y$. We know that X is Banach. Suppose Y is Banach, then by the $\boxed{\text{Open Mapping theorem}}$ or its corollary $\boxed{\text{Proposition 5.30}}$, we need the inverse of I to be bounded: specifically, can we find $c > 0$ such that $c\|f\|_u \le \|I(f)\|$ for all $f \in X$? This is impossible. We can demonstrate that by choosing a sequence of continuous functions f_n (for $n \ge 2$) that are triangles with base $[0, 2/n]$ and height 1 at $x = 1/n$, so that $\|f_n\|_u = 1$, but $\|f_n\| = 1/n$. This implies that $1/c \ge n$ for all n, which is impossible.

You don't have to use Open Mapping. You can directly find a sequence (f_n) that converges (under this norm) to a discontinuous function. For example, consider $f_n(x) = (1 - x)^n$; we know this converges pointwise, but not uniformly, to the discontinuous function $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$ on $[0, 1]$. In fact, it also converges to f with respect to this new norm, hence the space cannot be complete. To show that, we just need to show

$$\max_{x\in[0,1]} x(1 - x)^n \to 0$$

You can explicitly compute the max since it is a smooth function; by calculus, the derivative of $x(1 - x)^n$ is 0 when $x = 1/(n + 1)$, which gives a maximal value of $1/(n+1)(1-1/(n+1))^n$ (and the endpoints are not maximal, since they are 0). This maximal value is less than $1/(n + 1)$ since $(1 - 1/(n + 1))^n \le 1$, hence we can say $\|f_n - f\| \to 0$.

An even simpler example with the same discontinuous limit $f(x)$ is with $f_n(x) = 1/n$ (constant functions), as then you can see that $\|f_n - f\| = 1/n \to 0$.

Problem 4 Solution:

(a) Show that there is a bounded linear map $J : \mathcal{H} \to \mathcal{H}$ such that Jx is the unique element satisfying $\mathcal{A}(x, y) = \langle Jx, y\rangle$ for all $y \in \mathcal{H}$.

Solution: If x is fixed, then $y \mapsto \mathcal{A}(x, y)$ is linear (since $\mathcal{A}$ is bilinear), and it is bounded since we assume $|\mathcal{A}(x, y)| \le \beta\|x\|\cdot\|y\|$. Therefore, by the Riesz representation theorem, for a fixed x, we can write $\mathcal{A}(x, y) = \langle z, y\rangle$ for all y, for some $z \in \mathcal{H}$, and this z is unique. This z is some function of x, so let's write it as $z = J(x)$. Fixing y now, we know $x \mapsto \mathcal{A}(x, y)$ is linear in x (and bounded), and it is equivalent to writing $x \mapsto \langle J(x), y\rangle$. Therefore $J(x)$ must be bounded and linear.

(b) Show that $\alpha\|x\| \le \|Jx\|$.

Solution: If $x = 0$, this follows trivially, so assume $x \ne 0$ from now on. We assume $\alpha\|x\|^2 \le \|\mathcal{A}(x, x)\|$, and by part (a), $\mathcal{A}(x, x) = \langle J(x), x\rangle$. Now, using Cauchy-Schwartz, $|\langle J(x), x\rangle| \le \|J(x)\| \cdot \|x\|$, so combining this we have

$$\alpha\|x\|^2 \le \|\mathcal{A}(x, x)\| \le \|J(x)\| \cdot \|x\|$$

and dividing by $\|x\|$ gives the result.

(c) Show that J is bijective. *Hint: to show J is onto, first show that it has closed range*

Solution: First, show J has a trivial kernel. This follows immediately from what we proved in part (b). Similarly, J^* must have a trivial kernel, which we can show as follows: let $y \in \ker(J^*)$, so $J^*y = 0$, and hence for all x, $0 = \langle x, J^*y\rangle = \langle Jx, y\rangle$. In particular, we can choose $x = y$, so we have $0 = \langle Jy, y\rangle = \mathcal{A}(y, y)$. If $y \ne 0$, this violates our assumption that $\alpha\|y\|^2 \le \|\mathcal{A}(y, y)\|$.

Furthermore, the inequality in part (b) implies that J has closed range (via the open mapping theorem or its corollary Proposition 5.30 from Hunter & Nachtergaele).
Now, use the fact that for any $J \in \mathcal{B}(\mathcal{H})$, we have $\mathcal{H} = \overline{\text{ran}(J)} \oplus \ker(J^*)$ (cf. Theorem 8.17 in Hunter and Nachtergaele; note that the book suggests using the "projection theorem" for this proof, presumably meaning Thm. 6.13, but this approach does not seem as straightforward). Since we showed the range of J is closed and the kernel of J^* is trivial, we concluded that J is onto. We showed the kernel of J is trivial, so it is also one-to-one, and hence J is bijective.
Make sure not to claim that injective and closed range implies bijective; for example, the right-shift operator on $\ell^2(\mathbb{N})$ is injective and has closed range, but is not onto.

(d) Show that for any bounded linear functional $\varphi \in \mathcal{H}^*$, there exists a unique element $x \in \mathcal{H}$ such that $\mathcal{A}(x, y) = \varphi(y)$ for all $y \in \mathcal{H}$. *Note: this result is used to prove existence and uniqueness of weak solutions of PDE*

Solution: Again using the Riesz representation theorem, we can write $\varphi(y) = \langle z, y\rangle$ for some unique $z \in \mathcal{H}$. But we can also wrote $\varphi(y) = \mathcal{A}(x, y) = \langle Jx, y\rangle$ for some unique Jx. Hence $Jx = z$. Because J is bijective, the equation $Jx = z$ has a solution x (and it is unique).

Comment: we just proved exercise 12.23 in the Hunter & Nachtergaele book, which is used to prove Theorem 12.81, which is the famous "Lax-Milgram" theorem. To see how this is used to prove existence and uniqueness for weak solutions to PDE, see, e.g., http://drp.math.umd.edu/Project-Slides/DRP-Talk-Spring-2016.pdf.

Problem 5 Solution: It does converge. You could probably prove this via Fubini's theorem (with the counting measure, to turn the sum into an integral), but here's a method with the monotone convergence theorem ("MCT"). Define

$$f_n(x) = \begin{cases} 0 & x < n \\ f(x) & x \geq n. \end{cases}$$

Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_n^{\infty} f(x)\,dx &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_n^{\infty} f(x)\,dx \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \int_{\mathbb{R}} f_n(x)\,dx \\
&= \lim_{N\to\infty} \int_{\mathbb{R}} \sum_{n=1}^{N} f_n(x)\,dx \quad \text{(linearity of integral)} \\
&= \int_{\mathbb{R}} \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x)\,dx \quad \text{(this is monotone in } N \text{ due to non-negativity, so use } \boxed{\text{MCT}}\text{)} \\
&= \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)\,dx \\
&= \int_{\mathbb{R}} \lfloor x \rfloor f(x)\,dx \quad \text{(by observation)} \\
&\leq \int_{\mathbb{R}} x f(x)\,dx
\end{aligned}$$

so it is a bounded, monotone sequence, hence it converges.

You can also use Fubini's theorem, writing the sum as an integral via the counting measure (see example 12.30 in the book). Fubini applies, since both the counting measure and Lebesgue measure are σ-finite. Everything is non-negative, so we can ignore the absolute value. Thus Fubini says that if the integral works in either order, the integral exists, and it's the same value in either order. So you can jump straight to the last three lines above, and then since it is bounded, concluded the equality with the sum in the original order, and make the same conclusion.

You can also use DCT to interchange the limit, since $\sum_{n=1}^{N} f_n(x) \leq \lfloor x \rfloor f(x)$, and we know $xf(x)$ is integrable, so this is a dominating function.