Problem 1:
(a) Assume that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Suppose that for all $x, y \in \mathbb{R}^n$, defining the functions $g(t) = f(tx + (1-t)y)$ and $h(t) = tf(x) + (1-t)f(y)$, it holds that $(g-h)'$ is monotonically increasing for $t \in [0, 1]$. Prove that $f$ is convex, i.e., $g(t) \leq h(t) \quad \forall t \in [0, 1]$.

(b) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable, and suppose that on an open ball $U$ containing 0, we have $\nabla \cdot F = 0$.

1. Let $\varphi(x) = \int_0^x F \cdot dr$ for $x \in U$. We haven’t specified the path from 0 to $x$. Is $\varphi$ well-defined? Justify your answer.

2. Show that for arbitrary points $x$ and $y$ in $U$, $\int_y^x (\nabla \varphi) \cdot dr = \int_y^x F \cdot dr$. (This lets us conclude that $\nabla \cdot F = 0 \implies F = \nabla \varphi$; the converse is true as well, via direct calculation).

Problem 2: Prove that the Fredholm integral equation $x = Fx$ has unique continuous solution where

$$(Fx)(t) = \int_0^1 K(s, t, x(s)) \, ds + w(t), \quad t \in [0, 1], \quad w \in C([0, 1])$$

with $K$ continuous on $[0, 1] \times [0, 1] \times \mathbb{R}$ and $|K(s, t, \xi) - K(s, t, \eta)| \leq \theta|\xi - \eta|$ for some $0 < \theta < 1$.

Problem 3:
(a) Let $\mathcal{H} = L^2([0, 1])$ and define the operator $T : \mathcal{H} \to \mathcal{H}$ as

$$(Tf)(x) = x \cdot f(x)$$

1. Determine the point, continuous and residual spectrum of $T$ (with brief justification).

2. Is $T$ compact? Prove your answer.

(b) Consider the space $C([0, 1])$ with the norm $\|f\| = \sup_{x \in [0, 1]} |x \cdot f(x)|$. Is this a valid norm? If so, then is this space Banach? Prove your answers.

Problem 4:
Suppose $A : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bilinear form (that is, linear in each argument) on a Hilbert space $\mathcal{H}$, and there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \|x\|^2 \leq \|A(x, x)\|, \quad |A(x, y)| \leq \beta \|x\| \cdot \|y\| \quad \text{for all } x, y \in \mathcal{H}$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product.
(a) Show that there is a bounded linear map \( J : \mathcal{H} \to \mathcal{H} \) such that \( Jx \) is the unique element satisfying \( \mathcal{A}(x, y) = \langle Jx, y \rangle \) for all \( y \in \mathcal{H} \).
(b) Show that \( \alpha \|x\| \leq \|Jx\| \).
(c) Show that \( J \) is bijective. \textit{Hint: to show \( J \) is onto, first show that it has closed range}
(d) Show that for any bounded linear functional \( \varphi \in \mathcal{H}^* \), there exists a unique element \( x \in \mathcal{H} \) such that \( \mathcal{A}(x, y) = \varphi(y) \) for all \( y \in \mathcal{H} \). \textit{Note: this result is used to prove existence and uniqueness of weak solutions of PDE}

**Problem 5:** Let \( f \) be a measurable, non-negative function, and suppose \( \int_0^\infty x f(x) \, dx < \infty \). Determine if the following series converges:

\[
\sum_{n=1}^{\infty} \int_n^{n+1} f(x) \, dx.
\]