Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2018

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Assume that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Suppose that for all $x, y \in \mathbb{R}^n$, defining the functions $g(t) = f(tx+(1-t)y)$ and $h(t) = tf(x)+(1-t)f(y)$, it holds that $(g-h)'$ is monotonically increasing for $t \in [0,1]$. Prove that f is convex, i.e., $g(t) \le h(t)\ \forall t \in [0,1]$.

(b) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable, and suppose that on an open ball U containing 0, we have $\nabla \times \mathbf{F} = 0$.

(1) Let $\varphi(\mathbf{x}) = \int_0^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{x} \in U$. We haven't specified the path from 0 to $\mathbf{x}$. Is φ well-defined? Justify your answer.

(2) Show that for arbitrary points $\mathbf{x}$ and $\mathbf{y}$ in U, $\int_{\mathbf{y}}^{\mathbf{x}} (\nabla\varphi) \cdot d\mathbf{r} = \int_{\mathbf{y}}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{r}$. *(This lets us conclude that $\nabla \times \mathbf{F} = 0 \implies \mathbf{F} = \nabla\varphi$; the converse is true as well, via direct calculation).*

Problem 2: Prove that the Fredholm integral equation $x = Fx$ has unique continuous solution where

$$(Fx)(t) = \int_0^1 K(s,t,x(s))\,ds + w(t), \qquad t \in [0,1], \quad w \in \mathcal{C}([0,1])$$

with K continuous on $[0,1] \times [0,1] \times \mathbb{R}$ and $|K(s,t,\xi) - K(s,t,\eta)| \le \theta|\xi - \eta|$ for some $0 < \theta < 1$.

Problem 3:

(a) Let $\mathcal{H} = L^2([0,1])$ and define the operator $T : \mathcal{H} \to \mathcal{H}$ as

$$(Tf)(x) = x \cdot f(x)$$

(1) Determine the point, continuous and residual spectrum of T (with brief justification).

(2) Is T compact? Prove your answer.

(b) Consider the space $\mathcal{C}([0,1])$ with the norm $\|f\| = \sup_{x\in[0,1]} |x \cdot f(x)|$. Is this a valid norm? If so, then is this space Banach? Prove your answers.

Problem 4:

Suppose $\mathcal{A} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bilinear form (that is, linear in each argument) on a Hilbert space $\mathcal{H}$, and there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\|x\|^2 \le \|\mathcal{A}(x,x)\|, \quad |\mathcal{A}(x,y)| \le \beta\|x\| \cdot \|y\| \quad \text{for all } x, y \in \mathcal{H}$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product.

(a) Show that there is a bounded linear map $J : \mathcal{H} \to \mathcal{H}$ such that Jx is the unique element satisfying $\mathcal{A}(x, y) = \langle Jx, y \rangle$ for all $y \in \mathcal{H}$.
(b) Show that $\alpha \|x\| \leq \|Jx\|$.
(c) Show that J is bijective. *Hint: to show J is onto, first show that it has closed range*
(d) Show that for any bounded linear functional $\varphi \in \mathcal{H}^*$, there exists a unique element $x \in \mathcal{H}$ such that $\mathcal{A}(x, y) = \varphi(y)$ for all $y \in \mathcal{H}$. *Note: this result is used to prove existence and uniqueness of weak solutions of PDE*

Problem 5: Let f be a measurable, non-negative function, and suppose $\int_0^\infty x f(x)\, dx < \infty$. Determine if the following series converges:

$$\sum_{n=1}^{\infty} \int_n^\infty f(x)\, dx.$$