Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2017 (Draft v7, Aug 20)

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Let $\mathcal{F}$ be a family of equicontinuous functions from a metric space (X, d_X) to a metric space (Y, d_Y). Show that the completion of $\mathcal{F}$ is also equicontinuous.

(b) Let $(f_n)_{n\geq 1}$ be a sequence of functions in $\mathcal{C}([0,1])$. Let $||\cdot||$ be the sup norm. Suppose that, for all n, we have

- $||f_n|| \leq 1$,
- f_n is differentiable, and
- $||f_n'|| \leq M$ for some $M \geq 0$.

Show that the completion of $\{f_n\}_{n\geq 1}$ is compact, and therefore that it has a convergent subsequence.

Problem 2:

Show that there is a continuous function u on $[0,1]$ such that

$$u(x) = x^2 + \frac{1}{8}\int_0^x \sin(u^2(y))\,dy.$$

Problem 3:

Let $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n}$$

exists and equals $||f||_\infty$.

Problem 4:
Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\, dy.$$

This operator can be shown to be compact by using the Arzelà-Ascoli Theorem. For this problem, you may take compactness as fact.

(a) Find the adjoint operator K^* of K.
(b) Show that $||K||^2 = ||K^*K||$.
(c) Show that $||K|| = 2/\pi$. (Hint: Use part (b).)
(d) Prove that

$$K^n f(x) = \frac{1}{(n-1)!} \int_0^x f(y)(x-y)^{n-1}\, dy.$$

(e) Show that the spectral radius of K is equal to 0.
(*Hint: You might want to use the Stirling approximation bounds*

$$\sqrt{2\pi} n^{n+1/2} e^{-n} \le n! \le e\, n^{n+1/2} e^{-n}.)$$

Problem 5:

(a) State the Riesz Representation Theorem.

(b) Consider the second order boundary value problem

$$f''(x) = b(x)f(x) + q(x) \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad f'(0) = f'(1) = 0, \tag{1}$$

where $b, q \in C([0,1])$ are fixed and there exists a $\delta > 0$ such that $b(x) \ge \delta$ for $0 \le x \le 1$. A "weak solution" of (1) is a function f that makes the integro-differential equation

$$\int_0^1 (f'(x)c'(x) + b(x)f(x)c(x))\, dx = \int_0^1 q(x)c(x)\, dx \tag{2}$$

an identity for any $c \in C^1([0,1])$.
The goal of this problem is to show that there exists a unique solution f to (2) in the completion of $C^1([0,1])$.

(i) Define an inner product on $C^1([0,1])$ as

$$\langle g, h \rangle := \int_0^1 (g'(x)h'(x) + b(x)g(x)h(x))\, dx.$$

Verify that this is a valid real inner product.

(ii) Let $\mathcal{H}$ denote the completion of $C^1([0,1])$. Note that $\mathcal{H}$ is a Hilbert space with the inherited inner product

$$\left\langle \lim_m g_m, \lim_n h_n \right\rangle = \lim_{m,n} \langle g_m, h_n \rangle.$$

Define a functional $\varphi : \mathcal{H} \to \mathbb{R}$ by

$$\phi(u) := \int_0^1 q(x)u(x)\, dx.$$

Check that ϕ is bounded on $\mathcal{H}$. (Be clear about any norms you use.)

(iii) Conclude that there exists a unique $f \in \mathcal{H}$ that solves (1). (Explain!)

Problem 1 Solution:

(a) This part is almost trivial. It is just here to help with part (b).
Recall that $\mathcal{F}$ being equicontinuous means that, for any $\varepsilon > 0$, $\exists\ \delta > 0$ such that $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$ holds $\forall\ f \in \mathcal{F}$.
To show equicontinuity of the completion, we need only worry about the additional included functions. Let g be a function in the completion of $\mathcal{F}$ that was not in $\mathcal{F}$ to begin with. Since $\mathcal{F}$ is dense in the completion, we can find an $f \in \mathcal{F}$ that is arbitrarily close to g. In particular, chose $f \in \mathcal{F}$ such that $d_Y(f(x), g(x)) < \varepsilon/3$ $\forall x \in X$.
Let $\varepsilon > 0$. Note that

$$d_Y(g(x), g(y)) \leq d_Y(g(x), f(x)) + d_Y(f(x), f(y)) + d_Y(f(y), g(y)).$$

Since $f \in \mathcal{F}$, we can find a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon/3$ and we are done. This δ gives us $d_X(x,y) < \delta \Rightarrow d_Y(g(x), g(y)) < \varepsilon$.

(b) We will use the Arzelà-Ascoli Throrem: Let K be a compact metric space. A subset of $\mathcal{C}(K)$ is compact if and only if it is closed, bounded, and equicontinuous.
The completion of $\{f_n\}$, is, by definition, closed
By the assumptions of this problem, we also have that the completion of $\{f_n\}$ is bounded.
It remains to show that the completion of $\{f_n\}$ is equicontinuous.
Take $\varepsilon > 0$. Fix n. By the Intermediate Value Theorem, we know that, $\forall\ x, y \in [0,1]$, there exists a c between x and y such that $f_n(x) - f_n(y) = f_n'(c)(x-y)$.
Thus, we have that $f_n(x) - f_n(y) \leq M|x-y|$.
Define $\delta = \varepsilon/M$. We then have

$$|x-y| < \delta \quad \Rightarrow \quad |f_n(x) - f_n(y)| < \varepsilon.$$

Note that this is independent of the choice of n.
Thus, the family of functions $\{f_n\}$ is equicontinuous.
By part (a) we know then that the completion of this family is equicontinuous.
By the Arzelà-Ascoli Throrem, we then have that the completion of $\{f_n\}$ is compact, as desired.

Problem 2 Solution:

We will use the Contraction Mapping Theorem: If $T : X \to X$ is a contraction mapping on a complete metric space (X, d), then T has exaclty one fixed point. (i.e. There is exactly one $x \in X$ such that $T(x) = x$.)
Define

$$Tu(x) = x^2 + \frac{1}{8}\int_0^x \sin(u^2(y))\,dy.$$

Note that T maps $\mathcal{C}([0,1])$ functions to $\mathcal{C}([0,1])$ functions. Since $\mathcal{C}([0,1])$ is complete with respect to the sup norm $||\cdot||_\infty$, the contraction mapping theorem applies.
It remains to show that T is a contraction.

$$\begin{aligned}
||Tu - Tv||_\infty &= \sup_{0\leq x\leq 1} |Tu(x) - Tv(x)| \\
&= \sup_{0\leq x\leq 1} \left|\frac{1}{8}\int_0^x [\sin u^2(y) - \sin v^2(y)]\,dy\right| \\
&\leq \tfrac{1}{8} \sup_{0\leq x\leq 1} \int_0^x |\sin u^2(y) - \sin v^2(y)|\,dy
\end{aligned}$$

By the mean value theorem, we know that there is some $s \in [0,1]$ such that

$$\frac{\sin u - \sin v}{u - v} \leq \cos s \leq 1$$

so

$$|\sin u(y) - \sin v(y)| \leq |u(y) - v(y)|.$$

So, we have that

$$\begin{aligned}
||Tu - Tv||_\infty &\leq \tfrac{1}{8} \sup_{0 \leq x \leq 1} \int_0^x |u^2(y) - v^2(y)|\, dy \\
&= \tfrac{1}{8} \sup_{0 \leq x \leq 1} \int_0^x |u(y) + v(y)| \cdot |u(y) - v(y)|\, dy \\
&\leq \tfrac{1}{8} \sup_{0 \leq x \leq 1} \int_0^x (|u(y)| + |v(y)|) \cdot |u(y) - v(y)|\, dy
\end{aligned}$$

Since u and v are assumed to be continuous functions on the closed bounded interval $[0,1]$, they are bounded on $[0,1]$. Suppose that they are bounded by $M > 0$. Then

$$\begin{aligned}
||Tu - Tv||_\infty &\leq \tfrac{2M}{8} \sup_{0 \leq x \leq 1} \int_0^x |u(y) - v(y)|\, dy \\
&\leq \tfrac{M}{4} \int_0^1 |u(y) - v(y)|\, dy \leq \tfrac{M}{4} ||u - v||_\infty \int_0^1 \, dy \\
&= \tfrac{M}{4} ||u - v||_\infty
\end{aligned}$$

This may or may not be a contraction, depending on the value of M, but, we are trying to show **existence** of a solution in $\mathcal{C}([0,1])$. If we can show existence of a solution on some subset of $\mathcal{C}([0,1])$, we are done. So, let's limit our search to the set of continuous functions on $[0,1]$ that are bounded, in the uniform norm, by some **fixed** constant M such that $M < 4$. **Fix** such an M and define the space

$$C := \{u \in \mathcal{C}([0,1]) : ||u||_\infty \leq M\} \subseteq \mathcal{C}([0,1]).$$

Note that this is a closed (and non-empty!) subset of the complete $\mathcal{C}([0,1])$ and is therefore complete. Furthermore, M can be chosen so that $T : C \to C$.
Thus, we have a contraction maping on a complete space (that is a subspace of the space of interest). By the Contraction Mapping Theorem, there exists a unique fixed point $u \in C \subseteq \mathcal{C}([0,1])$, which is a solution to the problem.

Problem 3 Solution:
This is trivial if $||f||_\infty = 0$. So, let us consider the case where $||f||_\infty > 0$.
Note that

$$\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\, dx\right)^{1/n} \leq ||f||_\infty \left(\int_{\mathbb{R}} \frac{1}{1+x^2}\, dx\right)^{1/n} = ||f||_\infty \cdot \pi^{1/n} \to ||f||_\infty \tag{S1}$$

as $n \to \infty$.
On the other hand, by definition of $||f||_\infty$, for any $0 < \varepsilon < ||f||_\infty$, there exists an $A \subseteq \mathbb{R}$ (with positive Lebesgue measure) such that $|f(x)| > ||f||_\infty - \varepsilon \ \forall\ x \in A$.
Thus, we have

$$\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\, dx \geq \int_A \frac{|f(x)|^n}{1+x^2}\, dx \geq (||f||_\infty - \varepsilon)^n \int_A \frac{1}{1+x^2}\, dx.$$

Note that $\int_A \frac{1}{1+x^2}\, dx$ is strictly positive. Call it $c > 0$.

For all n, we now have

$$\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} \geq (||f||_\infty - \varepsilon)\cdot c^{1/n} \to ||f||_\infty - \varepsilon$$

as $n \to \infty$.
This implies that

$$\liminf_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} \geq ||f||_\infty - \varepsilon.$$

Since this holds for any $0 < \varepsilon < ||f||_\infty$, we have

$$\liminf_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} \geq ||f||_\infty.$$

On the other hand, (S1) implies that

$$\limsup_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} \leq ||f||_\infty.$$

Thus, we have

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n} = ||f||_\infty,$$

as desired.

Problem 4 Solution:
Note: This is the Volterra operator. Recall that $L^2([0,1])$ is a Hilbert space with inner product

$$< f, g >= \int_0^1 f(x)g(x)\,dx.$$

Note that, in what follows, the norms will be switching between the Hilbert space norm

$$||f|| = \sqrt{< f, f >}$$

and the operator norm

$$||K|| = \sup_{||f||\leq 1} ||Kf||.$$

(a) The adjoint K^* must satisfy

$$< f, Kg >=< K^*f, g >$$

for all $f, g \in L^2([0,1])$.
By inspection, it is easy to see that the adjoint will be defined by

$$K^*f(x) = \int_x^1 f(y)\,dy.$$

Check:

$$< f, Kg >= \int_0^1 f(x)(Kg)(x)\,dx = \int_0^1 f(x)\int_0^x g(y)\,dy\,dx \overset{Fubini}{=} \int_0^1 g(y)\int_y^1 f(x)\,dx\,dy =< K^*f, g > \quad \checkmark$$

(b) For any $||f|| \leq 1$,

$$||Kf||^2 =< Kf, Kf >=< K^*Kf, f > \overset{C.S.}{\leq} ||K^*Kf||\cdot||f|| \leq ||K^*K|| \leq ||K^*||\cdot||K||.$$

Thus, we have that $||K|| \leq ||K^*||$.
By a symmetric argument, $||K^*|| \leq ||K||$ and so $||K||^2 = ||K^*K||$.

(c) Since K^*K is compact and self-adjoint, the Spectral Theorem gives us that its norm is the magnitude of its largest eigenvalue. So, we proceed by finding the eigenvalues of K^*K.
Suppose that $K^*Kf = \lambda f$. Then
$$\lambda f''(x) = \frac{\partial^2}{\partial x^2} K^*Kf = \frac{\partial^2}{\partial x^2} \int_x^1 f(y) \int_0^y f(u)\, du\, dy = \cdots = -f(x).$$
To solve the resulting second order differential equation $\lambda f''(x) + f(x) = 0$, note that the characteristic equation is $r^2 + 1/\lambda = 0$. For simplicity, use $\omega^2 = 1/\lambda$. We then get that f must be of the form
$$f(x) = c_1 e^{1\omega x} + c_2 e^{-i\omega x}.$$
Compute
$$K^*Kf(x) = \int_x^1 f(y) \int_0^y f(u)\, du\, dy = \cdots$$
$$\cdots = \frac{1}{\omega^2} f(x) + \frac{1}{i\omega}(c_1 - c_2)x - \frac{1}{\omega^2}(c_1 e^{i\omega} + c_2 e^{-i\omega}) - \frac{1}{i\omega}(c_1 - c_2) \tag{3}$$
From the second term and the fact that $K^*Kf(x) = \lambda f(x) = (1/\omega)f(x)$, we must have $c_1 = c_2$. Plugging this back into (3), we have
$$K^*Kf(x) = \frac{1}{\omega^2} f(x) + -2c_1 \cos(\omega x).$$
Since this should equal $\lambda f = \frac{1}{\omega^2} f(x)$, we are forced to have the cosine term be zero and so we get
$$\omega = \frac{2n+1}{2}\pi \qquad \text{for any } n \in \mathbb{Z}.$$
Thus, the eigenvalues have the form
$$\lambda = \frac{1}{\omega^2} = \frac{4}{\pi(2n+1)^2}$$
which is largest when $n = 0$, making $\lambda = 4/\pi^2$ the largest eigenvalue.
By the spectral theorem, we then have that
$$||K||^2 = ||K^*K|| = \frac{4}{\pi^2}$$
which implies that
$$||K|| = \frac{2}{\pi},$$
as desired.

(d) This is straightforward induction on n.

(e) The spectral radius, $r(K)$, of a bounded linear operator K, may be computed as
$$r(K) = \lim_{n\to\infty} ||K^n||^{1/n}.$$
From part (d), we have that
$$||K^n|| \le \frac{1}{(n-1)!}.$$
Thus,
$$||K^n||^{1/n} \le \left(\frac{1}{(n-1)!}\right)^{1/n}$$
By the Stirling approximation bounds, we know that
$$(n-1)! \ge \sqrt{2\pi}(n-1)^{n-1/2}e^{n-1},$$

so we have that

$$[(n-1)!]^{1/n} \geq (\sqrt{2\pi})^{1/n}(n-1)^{1-1/(2n)}e^{1/n-1}$$

which goes to $-\infty$ as $n \to \infty$.
In conclusion, the spectral radius is

$$r(K) = \lim_{n\to\infty} ||K^n||^{1/n} = 0,$$

as desired.

Problem 5 Solution:

(a) If φ is a bounded linear functional on a Hilbert space $\mathcal{H}$, there is a unique vector $y \in \mathcal{H}$ such that $\phi(x) =< y, x >$ for all $x \in \mathcal{H}$.

(b) (i) Verifying inner product:
- Take any $\alpha_1, \alpha_2 \in \mathbb{R}$. By linearity of the integral, it is easy to see that

$$< g, \alpha_1 h_1 + \alpha_2 h_2 >= \alpha_1 < g, h_1 > +\alpha_2 < g, h_2 >$$

for $g, h_1, h_2 \in C'([0,1])$.
- Note that, for any $g \in C'([0,1])$,

$$\langle g, g\rangle = \int_0^1 \left[(g'(x))^2 + b(x)(g(x))^2\right] dx \geq 0 \tag{4}$$

since $b \geq 0$.
- Clearly, $g = 0 \Rightarrow < g, g >= 0$. On the other hand, $b \geq \delta > 0$ ensures that $< g, g >= 0 \Rightarrow g = 0$.

(ii) We wish to show that there exists a constant $M \geq 0$ such that

$$|\phi(u)| \leq M||u||$$

where $|\cdot|$ is the Euclidean norm and $||\cdot||$ is the Hilbert space norm defined as $||\cdot|| = \sqrt{\langle\cdot,\cdot\rangle}$.
First, we claim that

$$||u|| \geq \sqrt{\delta}||u||_2 \quad \text{for any } u \in \mathcal{H}, \tag{5}$$

where $||\cdot||_2$ denotes the L^2 norm. Indeed, by (4),

$$\begin{aligned} ||u||^2 &= \int_0^1 \left[(u'(x))^2 + b(x)(u(x))^2\right] dx \\ &\geq \int_0^1 b(x)(u(x))^2\, dx \geq \delta \int_0^1 (u(x))^2\, dx = \delta||u||_2^2. \end{aligned}$$

Now, by the Cauchy-Schwartz inequality and (5),

$$|\phi(u)| \leq ||q||_2\, ||u||_2 \leq \frac{||q||_2}{\sqrt{\delta}}||u|| \quad \text{for any } u \in \mathcal{H}.$$

Since q is continuous on $[0,1]$, $||q||_2$ must be finite. We therefore conclude that $|\phi(u)| \leq M\,||u||$ for all $u \in \mathcal{H}$, and so ϕ is bounded on $\mathcal{H}$.

(iii) From the Riesz Representation Theorem we know there must exist an $f \in \mathcal{H}$ satisfying $\phi(u) = \langle f, u\rangle$. This f is the solution to (2)!