Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2017 (Draft v7, Aug 20)

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Let $\mathcal{F}$ be a family of equicontinuous functions from a metric space (X, d_X) to a metric space (Y, d_Y). Show that the completion of $\mathcal{F}$ is also equicontinuous.

(b) Let $(f_n)_{n\geq 1}$ be a sequence of functions in $\mathcal{C}([0,1])$. Let $||\cdot||$ be the sup norm. Suppose that, for all n, we have

- $||f_n|| \leq 1$,
- f_n is differentiable, and
- $||f_n'|| \leq M$ for some $M \geq 0$.

Show that the completion of $\{f_n\}_{n\geq 1}$ is compact, and therefore that it has a convergent subsequence.

Problem 2:

Show that there is a continuous function u on $[0,1]$ such that

$$u(x) = x^2 + \frac{1}{8}\int_0^x \sin(u^2(y))\,dy.$$

Problem 3:

Let $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n}$$

exists and equals $||f||_\infty$.

Problem 4:
Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\, dy.$$

This operator can be shown to be compact by using the Arzelà-Ascoli Theorem. For this problem, you may take compactness as fact.

(a) Find the adjoint operator K^* of K.
(b) Show that $||K||^2 = ||K^*K||$.
(c) Show that $||K|| = 2/\pi$. (Hint: Use part (b).)
(d) Prove that

$$K^n f(x) = \frac{1}{(n-1)!} \int_0^x f(y)(x-y)^{n-1}\, dy.$$

(e) Show that the spectral radius of K is equal to 0.
(*Hint: You might want to use the Stirling approximation bounds*

$$\sqrt{2\pi} n^{n+1/2} e^{-n} \leq n! \leq e\, n^{n+1/2} e^{-n}.)$$

Problem 5:

(a) State the Riesz Representation Theorem.

(b) Consider the second order boundary value problem

$$f''(x) = b(x)f(x) + q(x) \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad f'(0) = f'(1) = 0, \tag{1}$$

where $b, q \in C([0,1])$ are fixed and there exists a $\delta > 0$ such that $b(x) \geq \delta$ for $0 \leq x \leq 1$. A "weak solution" of (1) is a function f that makes the integro-differential equation

$$\int_0^1 (f'(x)c'(x) + b(x)f(x)c(x))\, dx = \int_0^1 q(x)c(x)\, dx \tag{2}$$

an identity for any $c \in C^1([0,1])$.
The goal of this problem is to show that there exists a unique solution f to (2) in the completion of $C^1([0,1])$.

(i) Define an inner product on $C^1([0,1])$ as

$$\langle g, h \rangle := \int_0^1 (g'(x)h'(x) + b(x)g(x)h(x))\, dx.$$

Verify that this is a valid real inner product.

(ii) Let $\mathcal{H}$ denote the completion of $C^1([0,1])$. Note that $\mathcal{H}$ is a Hilbert space with the inherited inner product

$$\left\langle \lim_m g_m, \lim_n h_n \right\rangle = \lim_{m,n} \langle g_m, h_n \rangle.$$

Define a functional $\varphi : \mathcal{H} \to \mathbb{R}$ by

$$\phi(u) := \int_0^1 q(x)u(x)\, dx.$$

Check that ϕ is bounded on $\mathcal{H}$. (Be clear about any norms you use.)

(iii) Conclude that there exists a unique $f \in \mathcal{H}$ that solves (1). (Explain!)