**Applied Analysis Preliminary Exam** (Hints/solutions)
10.00am–1.00pm, August 18, 2016

**Problem 1:**

(a) Prove (using the comparison test or the Weierstrass-M test) the "Cauchy root test": If $C = \limsup_{n\to\infty} |a_n|^{1/n} < \infty$, then the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges uniformly if $|z| < 1/C$ and diverges if $|z| > 1/C$.

(b) What does this result say about the series

$$\sum_{n=0}^{\infty} 2^n \sin(n) z^n$$

*Solution/Hint*: Mainly straightforward.

**Problem 2:** (The two sub-problems are unrelated)

(a) One of the requirements of the Weierstrass Approximation Theorem is that the function to be approximated is continuous on a closed and bounded interval $I$. Show that the Approximation Theorem does not hold if we replace $I$ by a bounded open interval $(a, b)$ by showing that if $f(x) = 1/(b - x)$, then $f : (a, b) \to \mathbb{R}$ cannot be uniformly approximated by polynomials.

*Solution/Hint*: Mainly straightforward. Do not use Taylor series; even if the Taylor series doesn't approximate the function, that doesn't prove that there is no other polynomial which approximates the function.

(b) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}$ a bounded linear operator. If

$$\lim_{n\to\infty} \sup_{\substack{u \perp \{e_1,\ldots,e_n\} \\ u \neq 0}} \frac{\|Au\|}{\|u\|} = 0$$

then prove $A$ is a compact operator.

*Solution/Hint*: The general idea is to approximate $A$ with a finite-rank operator, which is therefore compact, and then show that these approximations converge uniformly to $A$. If you do this, be careful if you use double-indices.

Another approach is to show that $A$ maps weakly convergent sequences to strongly convergent sequences. With this approach, be careful with what you show. Do not prove that if $x_n \rightharpoonup x$ that then $Ax_n \to x$, unless you prove this for *any* sequence $(x_n)$; several students did this for a *particular* sequence $(x_n)$, but this is not strong enough to conclude $A$ is compact.

**Problem 3:** Let $(X, d)$ be a complete metric space. A function $f : X \to X$ is said to be a contraction if there exists $c < 1$ such that $d(f(x), f(y)) \le c \cdot d(x, y)$ for all $x, y \in X$. A function $f$ is said to be a *weak contraction* if

$$d(f(x), f(y)) < d(x, y) \quad \forall\, x \neq y,\ x, y \in X.$$

Note that a weak contraction is Lipschitz continuous with Lipschitz constant 1.

(a) Prove the following variant of the contraction mapping theorem: if $f$ is a *weak contraction* and the space $X$ is *compact*, then $f$ has a unique fixed point in $X$. Hint: consider the function $g(x) = d(x, f(x))$ over $X$.

*Solution/Hint*: Using the hint, there is a very short proof. The uniqueness argument is similar to the argument used in the standard contraction mapping theorem (most students got this part).

Overall, you do not want to explicitly construct a sequence $(x_n)$ as in the standard theorem. Some students did this and showed $g(x_n) \to 0$, and then (falsely) concluded that $x_n \to 0$.

Several styles of attempted proof were not correct because they would have worked had $f$ been non-expansive, e.g., $d(f(x), f(y)) \le d(x, y)$ ($\le$ instead of $<$), for which the result isn't true (e.g., $f(x) = -x$ on $X = [-2, -1] \cup [1, 2]$ has no fixed point). For example, consider approximating $d$ by some $d_n$ which is contractive in the usual sense (for some $L_n < 1$ and $L_n \to 1$). A problem here is that this can change the range; we need that $f(X) \subset X$, and this fails for $X = [-2, -1] \cup [1, 2]$ if you replace $f$ with a contractive $f_n$, so you would need to prove why you preserve the range/domain.

If you can't figure out the hint, then the solution is below (hold it up to a mirror to read):

*Solution/Hint*: Define $g(x) = d(x, f(x))$. Then $f$ is continuous (since it is Lipschitz continuous), and so is the metric (by the triangle inequality), hence $g$ is continuous and therefore it achieves its minimum over the compact set $X$. If this minimum is zero (it cannot be lower, due to the non-negativity) then by properties of the metric, this means there is some $x$ such that $x = f(x)$, i.e., there is a fixed point. If the minimum is non-zero, which is achieved at $x$, then we have a contradiction since $d(f(x), f^2(x)) < d(x, f(x))$ so it was not actually a minimum. Uniqueness follows the same was as the standard theorem: if $x = f(x)$ and $y = f(y)$ then either $x = y$ or $d(x, y) = d(f(x), f(y)) < d(x, y)$, which is a contradiction.

(b) Let $X$ be $\ell^2(\mathbb{N})$ and $f(x) = L(x) + b$ be an affine function defined by mapping $L : x \mapsto y$ where $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ with $y_n = (1 - \frac{1}{n})x_n$, and $b = (\frac{1}{n})_{n\in\mathbb{N}}$. Prove or disprove that $f$ has a fixed point in $X$. Does your answer change if $X$ is the closed unit ball in $\ell^2(\mathbb{N})$?

*Solution/Hint*: No, it does not. If it did, you can see that you solve the equation $x = L(x) + b$ or $(I - L)x = b$ which means that, term-by-term, you have $1/nx_n = b_n = 1/n$, so $x_n = 1$, but $(1, 1, 1, \dots) \notin \ell^2(\mathbb{N})$ since it is not square-summable.

If $X$ is the closed unit ball, we still do not have a fixed point, for any of these reasons: (1) it is a smaller set, so if we didn't have a fixed point in $\ell^2$ we can't have one in a subset; (2) $f$ does not map $X$ to $X$, since it keeps shifting out (due to the offset term). Our the work from part (a) does not help us since the closed unit ball in $\ell^2$ is not compact (it is bounded but not totally bounded), so we can't use that to make a conclusion either way.

**Problem 4:** Let $\mathcal{H}$ be an infinite dimensional Hilbert space, and $M$ a subspace of $\mathcal{H}$. If $\varphi \in M^*$, prove $\varphi$ has a *unique* norm-preserving extension to a bounded linear functional on all of $\mathcal{H}$, and that this extension vanishes on $M^\perp$.

*Solution/Hint*:

We could apply Hahn-Banach directly but it's not powerful enough. This is a dead end. We must use the fact that we are in a Hilbert space, not just a normed vector space. For example, in a Banach space such as $X = (C([0, 1]), \|\cdot\|_\infty)$, if we have a subspace such as the periodic functions $M = \{f \in X \mid f(0) = f(1)\}$, consider a functional $\phi(f) = f(0)$ defined on $M$. We could extend this to $\psi(f) = f(0)$, or we could extend it to $\psi(f) = f(1)$; both extensions agree on $M$, and both have norm 1. So Hahn-Banach by itself is not enough.

Instead, try using the BLT theorem (why? well, is $M$ necessarily closed?) and then the Riesz-Representation theorem. To show it is unique, decompose $\mathcal{H}$ into subspaces and use the fact that the norms cannot increase.

**Problem 5:** State and prove Fatou's Lemma. Give an example to show why in general we must have an inequality and not an equality.

*Solution/Hint*: Try using the Monotone Convergence Theorem.