Applied Analysis Preliminary Exam: Solutions/Hints
10.00AM–1.00PM, AUGUST 18, 2015

Hints for Problem 1: See the book. Note that (b) and (c) are equivalent on the real line — does this imply they must be equivalent on an arbitrary Banach space?

Hints for Problem 2: See the book. As the syllabus for this exam does not cover non-separable Hilbert spaces, you were allowed to assume the Hilbert space was separable.

Hints for Problem 3:

(a) Don't forget to provide counter-examples if you claim something like $L^1(\mathbb{T}) \not\subset L^2(\mathbb{T})$. The answer is $H^1(\mathbb{T}) \subsetneq L^2(\mathbb{T}) \subsetneq L^1(\mathbb{T})$, which you should be able to prove (the Sobolev embedding theorem can help). The fact that L^1 is the biggest makes the subsequent parts of this problem simpler.

(b) Don't confuse Fourier series with the Fourier transform. Note that the book discusses Fourier series mainly for L^2 (which includes H^1). For L^1, the Fourier coefficients are certainly well-defined; we also have a reconstruction formula for L^1 but the partial sum of Fourier coefficients does not converge pointwise a.e. nor does it converge in the L^1 norm, though it does converge with respect to other senses, so this part of the question was graded leniently due to the ambiguity.

(c) See the book.

(d) You only need to do this for L^1 since it contains the other spaces; use Hölder's inequality.

(e) Hint: use properties of orthonormal bases.

(f) By itself, H^1 is complete with respect to its own norm, but is it closed with respect to the L^2 norm? Could it be closed *and* dense? (is it dense in L^2?) Most students who answered this question correctly drew a picture to help clarify their thoughts.

Hints for Problem 4: For (a), try using $\mathcal{H} = \ell^2(\mathbb{N})$ for your example; or, try using example 9.5. Note that you must be in infinite dimensions, otherwise all linear operators are compact since they have finite rank. Part (b) has several proofs, most of which are good to discover yourself (and some of which can be very short). An interesting approach is to use the fact that compact operators turn weak convergence into strong convergence, and then use the Banach-Alouglu theorem (but there are solutions that do not use Banach-Alouglu).

Another hint for (b): some students tried to use *contraction* ideas, but this is a red herring. By rescaling the linear operator, you could just as well consider A such that $\|Ax\| < 2\|x\|$ for all $x \in \mathcal{H} \setminus \{0\}$ and then prove that if A is compact, $\|A\| < 2$. There is nothing special about the actual value, only about the $\leq$ vs $<$.

Some other mistakes were using "facts" like $a_n < b_n$ implies $\lim a_n < \lim b_n$ or $\sup a_n < \sup b_n$. This is not true; when you take the limit or supremum, you must change $<$ to $\leq$.

Hints for Problem 5:

The heart of the problem is *showing* that for the Lebesgue integral,

$$\int_{\mathbb{R}} f(x)\, d\mu = \lim_{t\to\infty} \int_{-t}^{t} f(x) d\mu$$

(or any variant for an interval like $[0, \infty)$). For the Riemann integral, this equality is just an identity, since the left-hand-side is defined as the improper integral on the right side. For the Lebesgue integral, you should prove these are equivalent (hint: use the monotone or dominated convergence theorems).

It was not necessary to define the Riemann integral using Riemann sums, nor to use the definition of the Lebesgue integral in terms of simple functions. The main tools to use are the fact that if both types of integrals exist, then they must have the same value; and that if the Riemann integral exists, then so does the Lebesgue integral.

Remark: take a function like $f(x) = \sin(x)/x$. This has an improper Riemann integral over $\mathbb{R}$ since the positive and negative terms cancel, but if we take the absolute value, it is no longer Riemann integrable since we can lower-bound the integral with the harmonic series. This problem implies that therefore it is not Lebesgue integrable. This agrees with with we know, since if it were Lebesgue integrable, that means it is in $L^1(\mathbb{R})$, and therefore its Fourier transform, which is the discontinuous box-car function, should be continuous (via Riemann-Lebesgue), which is a contradiction.

Remark: the monotone convergence theorem is *not* valid for Riemann integrals.