Applied Analysis Preliminary Exam
10.00AM–1.00PM, AUGUST 18, 2015

INSTRUCTIONS. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do *not* wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student number on your exam. Each problem is worth 20 points.

Problem 1: Let $\{x_\alpha \in X \mid \alpha \in I\}$ be an indexed set in a Banach space X, where the index set I may be countable or uncountable. For each finite subset J of I, we define the partial sum S_J by

$$S_J = \sum_{\alpha \in J} x_\alpha.$$

The *unordered sum* of the indexed set $\{x_\alpha \mid \alpha \in I\}$ is said to *converge unconditionally* to x if for every $\epsilon > 0$, there is a finite subset $J^\epsilon \subset I$ such that $\|S_J - x\| < \epsilon$ for all finite subsets J that contain J^ϵ.

Consider the following types of convergence of a series (with a countable index) in the Banach space X:

(a) convergence

(b) unconditional convergence

(c) absolute convergence

(d) Cauchy convergence (in the usual sense; you do not need to consider the sense of Cauchy convergence for an *unordered sum*)

What are the causal relationships among these four types of convergence? Your answer should cover all possible pairwise relationships among the different types of convergence, and each relationship should be of the form $(f) \implies (g)$, $(g) \implies (f)$, $(f) \iff (g)$, or "no implication". Briefly justify each, e.g., if your answer is $(f) \implies (g)$, in addition to proving the implication, you should also briefly justify why (g) does not imply (f) [using, for example, a counter-example].

Hint: for one of the counter-examples, consider a special case when (x_n) are orthogonal.

Problem 2: Theorem 6.26 in Hunter and Nachtergaele's book gives 5 equivalent ways to define the completeness of an orthonormal set in a Hilbert space, and thereby defines an orthonormal basis.

(a) State three definitions of an orthonormal basis in a Hilbert space (if more than three definitions are given, the first three will be graded)

(b) Prove at least two implications from among your three definitions, e.g., Definition 1 $\implies$ Definition 2 $\implies$ Definition 3. They need not be $\iff$, and you can choose the order.

Problem 3: Every Banach space is defined by a norm that identifies the set of functions that are bounded in that norm. Consider the set of functions that are bounded in the $L^1(\mathbb{T})$ norm $\|\cdot\|_{L^1}$, a (possibly different) set of functions that are bounded in the $L^2(\mathbb{T})$ norm $\|\cdot\|_{L^2}$, and a (possibly different) third set that are bounded in the Sobolev $H^1(\mathbb{T})$ norm $\|\cdot\|_{H^1}$, where $\mathbb{T}$ is the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

We denote the above sets as $L^1(\mathbb{T}), L^2(\mathbb{T})$ and $H^1(\mathbb{T})$, respectively, and their corresponding Banach spaces as $\left(L^1(\mathbb{T}), \|\cdot\|_{L^1}\right), \left(L^2(\mathbb{T}), \|\cdot\|_{L^2}\right)$ and $\left(H^1(\mathbb{T}), \|\cdot\|_{H^1}\right)$, respectively.

(a) As sets, how are $L^1(\mathbb{T}), L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ related in terms of inclusion, e.g., is $L^1 \subset L^2$, etc.? Briefly justify each inclusion or provide a counter-example if there is no inclusion.

(b) For which of the sets ($L^1(\mathbb{T}), L^2(\mathbb{T})$ and $H^1(\mathbb{T})$) are Fourier series defined? What is the formula for the Fourier series coefficients (using either coefficients a_n and b_n for sine/cosine series, or c_n for exponentials) and what is the formula for writing a function in terms of its Fourier coefficients?

(c) State a definition of $H^1(\mathbb{T})$.

(d) For each space, prove that the Fourier coefficients are uniformly bounded.

(e) Prove that for all $f \in L^2(\mathbb{T})$, the Fourier coefficents decay to 0 as n grows (that is, as $n \to \infty$ for sine/cosine series, or as $|n| \to \infty$ for exponentials).

(f) As a set, is $H^1(\mathbb{T})$ a closed subset of $L^2(\mathbb{T})$? Justify your answer.

Problem 4: Let $\mathcal{H}$ be a Hilbert space, and $A : \mathcal{H} \to \mathcal{H}$ be a self-adjoint bounded linear operator such that $\|Ax\| < \|x\|$ for all $x \in \mathcal{H} \setminus \{0\}$.

(a) Give an example of such an operator A such that $\|A\| = 1$

(b) Prove that if we additionally assume that A is compact, then $\|A\| < 1$.

Problem 5: Suppose f is Riemann integrable over an infinite interval of the real line (such an integral can only exist in the improper sense). Prove that f is Lebesgue integrable over the same interval if and only if the improper integral converges absolutely, e.g., $\lim_{t\to\infty} \int_{-t}^{t} |f(x)|\,dx < \infty$ or $\lim_{t\to\infty} \int_0^t |f(x)|\,dx < \infty$.