Solutions

Problem 1:

(a) What does it mean for an operator to be compact? A linear operator $T : H \to H$ is compact if $T(B)$ is a precompact subset of H for every bounded subset $B \subset H$ (recall "precompact" means its closure is compact, or equivalently, that every sequence has a convergent subsequence). That is to say for every bounded sequence $(x_n) \subset H$, then (Tx_n) has a convergent subsequence.

(b) Discuss convergence: Note that the problem doesn't ask the student to prove if the limit is in $\mathcal{B}(H)$, so this may be assumed.

(a) We show convergence in norm is sufficient.

Solution 1 Let $(x_m) \subset H$ be a bounded sequence with $\|x_m\| \leq B$ for all m. We will show that there is a subsequence (m_k) such that (Ax_{m_k}) is Cauchy, and since H is complete, therefore it is convergent. The only tricky part is defining m_k. Since A_1 is compact, there is a subsequence $(m_{k(1)})$ such that $A_1(x_{m_{k(1)}})$ is convergent (to, say, y_1). Since A_2 is compact, there is a subsequence $m_{k(2)}$ **of** $(m_{k(1)})$ such that $A_2(x_{m_{k(2)}})$ is convergent to y_2 (and $A_1(x_{m_{k(2)}})$ is still convergent to y_1, since this is a subsequence of the subsequence).

For each k, we have a subsequence of the subsequence associated with $k-1$. We can take the k^{th} term of this new subsequence, and make this into a master subsequence (m_k). This is known as the **diagonalization trick**. Since this master subsequence is bounded, and $\|A_n - A\| \to \infty$, an $\epsilon/3$ argument shows that the sequence (y_k) is Cauchy, and thus there is some y with $y_k \to y$, and then again using an $\epsilon/3$ argument we see that $Ax_{m_k} \to y$, thus proving that A is a compact operator.

Solution 2 A slicker proof is using the fact that a compact operator can be arbitrarily well-approximated by a finite-rank operator; using this, the proof is trivial (basically, that's what this problem is trying to show).

Solution 3 Use the fact that a compact operator (on a Hilbert space) maps weakly convergent sequences to strongly convergent ones, i.e., if A_n is compact, then $x_k \rightharpoonup x$ implies $A_n x_k \to A_n x$. Thus we only need to show $Ax_k \to Ax$. We do this with the usual triangle inequalities:

$$\|Ax_k - Ax\| \leq \|Ax_k - A_n x_k\| + \|A_n x_k - A_n x\| + \|A_n x - Ax\|$$

and we can make all terms small. But note that we require norm convergence and boundedness in order for the first and third terms to be BOTH small. If we have only strong convergence, then we can make them small separately (by choosing n large enough) but not necessarily have both of them small. The middle term is arbitrarily small by choosing k sufficiently large.

Solution 4 Let $B \subset H$ be bounded, so for every n, $A_n(B)$ is pre-compact and hence totally bounded. It is sufficient to show $A(B)$ is totally bounded since H is Banach. Let the radius of B be M, i.e., $x \in B$ implies $\|x\| \leq M$. We will show that for any $\epsilon > 0$, there is a finite ϵ-net of $A(B)$. Fix $\epsilon > 0$.

Pick n large enough such that $\|A_n - A\| < \epsilon/(3M)$. Since $A_n(B)$ is totally bounded, let $(y_i)_{i=1}^N \subset B$ be a finite $\epsilon/3$ net of $A_n(B)$. Since $y_i \in A_n(B)$, we can write it as $A_n x_i$.

For any $x \in B$, we have

$$\|(A_n - A)x\| < \epsilon/(3M)\|x\| \leq \epsilon/3.$$

Hence if we pick an arbitrary point $A(x) \in A(B)$, it is within $\epsilon/3$ of the point $A_n(x) \in A_n(B)$. By the triangle inequality, since (x_i) is an $\epsilon/3$ net for $A_n(B)$, there is some Ax_i that is within ϵ of $A(x)$.

Explicitly, for $x \in B$, there is some x_i such that

$$\begin{aligned}\|Ax - Ax_i\| &\leq \|Ax - A_n x\| + \|A_n x - A_n x_i\| + \|A_n x_i - Ax_i\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

Hence $\{Ax_i\}$ is a finite ϵ-net for $A(B)$, and since ϵ was arbitrary, this means $A(B)$ is totally bounded, hence pre-compact.

(b) We show strong convergence is not sufficient. Take A_n to be defined as in Example 5.46 in the book, where for $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \subset \ell^2$ we have $A_n(x) = (x_1, \ldots, x_n, 0, 0 \ldots)$. For any fixed n, $\lim_n A_n x = x$ therefore A_n converges strongly to $A = I$ (note that it does not converge in norm, since $\|A_n - A_m\| = 1$ for $m \neq n$). The identity is not compact. For example, take a sequence (x_m) where $x_m = (0, \ldots, 1, 0, \ldots)$, i.e., it is 0 except for a 1 in the m^{th} position. This sequence is bounded but since $\|x_m - x_{m'}\| = 1$ for all $m \neq m'$, it cannot have a Cauchy subsequence, hence it does not have a convergent subsequence, so the identity is not compact. (Note: in finite dimensions, the identity is compact).

(c) Since strong convergence implies weak convergence, weak convergence is not sufficient

Problem 2: First, we note that we can calculate the Fourier series and make coefficients $\hat{u}_n$, which are uniquely determined by the problem specifications *except* for $\hat{u}_0$. We do the calculation using integration-by-parts twice (though this fails for $n = 0$), or by directly calculating $\hat{u}''$ and then divide by $(in)^2$.

- Is the function uniquely specified? No, because $\hat{u}_0$ is not determined. There can be an arbitrary constant offset (but no arbitrary linear offset, in order to keep it periodic — this should be mentioned).
- The Fourier coefficients are of the form $\hat{u}_n = c/n^3$ where c is an unimportant constant. A function u is in the Sobolev space H^s iff $n^s \hat{u}_n \in \ell^2$, so this means we need $cn^{s-3} \in \ell^2$. From calculus, we know that $n^{-.5-\epsilon} \in \ell^2$ iff $\epsilon > 0$, so $u \in H^s$ for $s < 2.5$. Thus s is not unique (note that $H^s \subset H^{s'}$ if $s' \leq s$, so s is never unique unless it is $s = 0$).

  Another way that is not quite as sharp: note that u'' is discontinuous, so therefore u'' cannot be in H^s for $s > 1/2$ otherwise we violate the Sobolev embedding theorem. Hence u cannot be in H^s for $s > 2.5$. It's also clear that u'' is a weak derivative of u so u is in H^s for $s \leq 2$. The region between 2 and 2.5 is not made clear with this method.

Problem 3:

(a) The question is whether we can interchange the limit and the integral. First, we observe that the integrand is bounded by one; there are several ways to prove this, e.g., the binomial theorem, or via induction with k. Now that the integrand is positive and bounded by 1, we can apply the Lebesgue Dominated Convergence Theorem and interchange the limit and integral.

Now, to evaluate the limit of the integrand, use standard techniques (e.g., L'Hôpital's rule) to get a value of 0 for $x \in (0, 1]$ and 1 for $x = 0$. Integrating this function gives a value of 0.

(b) The partial sums s_n are monotone since b_k and r are nonnegative. The partial sums are also bounded, since (b_k) is bounded (say, $b_k \le M$ for all k), and $r < 1$, so that

$$s_n \le M \sum_{k=1}^{n} r^k = \frac{Mr(1-r^n)}{1-r} \le \frac{Mr}{1-r}$$

Thus we have a bounded, monotone sequence of real numbers, so the Monotone Convergence Theorem says this sequence must converge. (Note that it need not converge to $Mr/(1-r)$, since M was just a bound on (b_k); rather, it converges to $r/(1-r) \cdot \limsup_k b_k$).

Problem 4:

(a) $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, and $H_3(x) = 8x^3 - 12x$.

(b) Follow the hint and let $v(x) = e^{-x^2}$, so the term in the hint is (where $v^{(m)}$ is the m^{th} derivative of v)

$$\begin{aligned}(-1)^n \int_{\mathbb{R}} H_n(x) v^{(m)}\, dx &= H_n(x) v^{(m-1)}\Big|_{\mathbb{R}} - \int_R 2n H_{n-1}(x) v^{(m-1)}\, dx \\ &= -\int_R 2n H_{n-1}(x) v^{(m-1)}\, dx \\ &= \ldots \\ &= (-1)^n 2^n n! \int_R H_0(x) v^{(m-n)}\, dx\end{aligned}$$

and $H_0(x) = 1$. If $n < m$, integrating once more gives 0 since v and its derivatives approach zero as x goes to $\pm\infty$, and this proves the orthogonality.

(c) This follows directly from part (b), since we have just moved the weight function to φ.

(d) Because this is an orthonormal basis, we just calculate

$$f_8 = \int_{\mathbb{R}} f(x) c_8 \varphi_8(x)\, dx.$$

Problem 5:

(a) Let $0 \in \operatorname{int} C$ and $x \in X$. Then there is an $\epsilon > 0$ such that $B_\epsilon(0) \subset C$, and in particular $\frac{\epsilon}{2} \in C$, so $\gamma_C(x) \le 2/\epsilon < \infty$. Now let C be convex, and let $x, y \in C$ with $\gamma_C(x) = \lambda$ and $\gamma_C(y) = \mu$. Then $x' = x/\lambda \in C$ and $y' = y/\mu \in C$, and by convexity, $z = \frac{\lambda}{\lambda+\mu} x' + \frac{\mu}{\lambda+\mu} y' \in C$. Since $z = (x+y)/(\lambda+\mu)$, it follows $\gamma_C(x+y) \le \lambda + \mu$. It helps to draw a picture to see how the sub-additive property fails if C is not convex.

(b) This follows immediately by defining the sub-linear functional $p(x) = \|\psi\|\|x\|$.

(c) Without loss of generality, shift C and d such that $0 \in \operatorname{int} C$, since the problem is translation invariant. Let $Y = \operatorname{span}(d)$ and define the linear functional $\psi(\lambda d) = \lambda\ \forall \lambda d \in Y$, so in particular $\psi(d) = 1$. Let $p(d) = \gamma_C(d)$ which is sub-linear, and $p(d) \ge 1$ since $d \notin C$. It follows $\psi(\lambda d) \le p(\lambda d)\ \forall \lambda z \in Y$ since either $\lambda > 0$ in which case we use the fact that both ψ and p are positive homogenous, or $\lambda \le 0$ in which case $\psi(\lambda d) \le 0 \le p(\lambda d)$.

We can apply the Hahn-Banach theorem and extend ψ to a linear functional Ψ on X, with $\Psi(x) \le p(x)\ \forall x \in X$. We still have $\Psi(d) = 1$. Since $p(x) \le 1\ \forall x \in C$, this implies

$\Psi(x) \leq \Psi(d)\ \forall x \in C$, and thus the hyperplane defined by $\{x \in X : \Psi(x) = \Psi(d)\}$ separates d and C.