Applied Analysis Preliminary Exam

10.00am–1.00pm, August 18, 2014

Instructions: You have three hours to complete this exam. Work all five problems. Each problem is worth 20 points. Please write on only one side of the paper, start each problem on a new page, and put your student ID number in the upper right hand corner of each page. For each problem, you must prove your conclusions or provide a counter-example.

Problem 1: Let H be an infinite dimensional Hilbert space. Recall that we say that a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges *weakly* to an operator $A \in \mathcal{B}(H)$ if

$$\lim_{n\to\infty} \langle A_n x,\, y\rangle = \langle Ax,\, y\rangle, \qquad \forall\, x, y \in H,$$

that $(A_n)_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges *strongly* to an operator A if

$$\lim_{n\to\infty} ||A_n x - Ax|| = 0, \qquad \forall\, x \in H,$$

and that $(A_n)_{n=1}^{\infty}$ converges *in norm* to A if

$$\lim_{n\to\infty} ||A_n - A|| = 0.$$

(a) Define what it means for an operator in $\mathcal{B}(H)$ to be *compact.*

(b) Suppose $(A_n)_{n=1}^{\infty}$ is a sequence of compact operators that converges to a limit A. For which of the three modes of convergence listed, if any, must A be compact? If any mode of convergence is not sufficient for A to be compact, provide a counterexample. Carefully motivate your statements.

Problem 2: Set $I = [-\pi, \pi]$, and suppose that $u :\ I \to \mathbb{C}$ is a continuously differentiable function. Suppose further that both u and its derivative are periodic, so that $u(\pi) = u(-\pi)$ and $u'(\pi) = u'(-\pi)$, and that its second derivative exists almost everywhere and satisfies

$$u''(x) = \begin{cases} 1 & \text{when } |x| < \pi/2, \\ -1 & \text{when } |x| > \pi/2. \end{cases}$$

Do the conditions specified uniquely determine u? Do they uniquely determine for which s the function u belongs to the Sobolev space $H^s(I)$?

Problem 3: Two unrelated limits.

(a) Find the following limit:

$$\lim_{k\to\infty} \int_0^1 \frac{1 + kx^2}{(1 + x^2)^k} dx$$

(b) Let $\{b_n\}$ be a bounded sequence of nonnegative numbers and let $r \in [0, 1)$. Define $s_n = \sum_{k=1}^{n} b_k r^k$ for $n = 1, 2, 3, \ldots$. Discuss the convergence/divergence of $\{s_n\}$.

Problem 4: Hermite polynomials

(a) Define the set of *Hermite polynomials*, $\{H_n(x)\}$, by

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n (e^{-x^2}), \text{ for } n = 0, 1, 2, 3, \ldots$$

Find $H_n(x)$ explicitly for $n = 0, 1, 2, 3$.

(b) *Fact* (no work required by you): $H_n'(x) = 2nH_{n-1}(x)$. Now prove that these polynomials satisfy a weighted orthogonality relation on $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\, dx = 0 \quad \text{for } n \neq m$$

Hint: This condition can be rewritten as

$$(-1)^m \int_{-\infty}^{\infty} H_n(x) \left(\frac{d^m e^{-x^2}}{dx^m}\right) dx = 0$$

For $m > n$, integrate by parts n times and evaluate the resulting integrals.

(c) Define

$$\varphi_n(x) := e^{-x^2/2} H_n(x) \quad \text{for } n = 0, 1, 2, \ldots$$

Prove that the functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ are mutually orthogonal in $L^2(-\infty, \infty)$.

(d) *Fact* (no work required by you): These orthogonal functions can be normalized. Therefore, assume that for a specific set of constants $\{c_n\}$, $\{c_n\varphi_n(x)\}_{n=0}^{\infty}$ form an orthonormal set in $L^2(-\infty, \infty)$. Now let $f(x)$ be an L^2 function. If

$$f(x) = \sum_{n=0}^{\infty} f_n\{c_n\varphi_n(x)\}$$

find an explicit formula for the coefficient f_8.

Problem 5: Let X be a normed linear space. A subset $C \subset X$ is called *convex* if $\forall\, x, y \in C$ and $\forall 0 \leq \lambda \leq 1$ then $\lambda x + (1 - \lambda)y \in C$. A functional f on X is *sub-additive* if $f(x + y) \leq f(x) + f(y) \forall\, x, y \in X$, and it is *sub-linear* if it is sub-additive and positive homogenous. A *hyper-plane* in $\mathbb{R}^n$ defined by a normal vector $a \in \mathbb{R}^n$ and offset $\beta \in \mathbb{R}$ is the set $\{x \in \mathbb{R}^n : \langle a, x\rangle = \beta\}$. A hyper-plane in X defined by a linear functional ϕ and offset β is the set $\{x \in X : \phi(x) = \beta\}$.

(a) The gauge function of a set C is defined $\gamma_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}$. Clearly γ_C is positive homogenous. Show that if $0 \in \text{int}C$ then γ_C is finite, and if C is convex then γ_C is sub-additive.

(b) A slightly stronger version of the Hahn-Banach theorem, in comparison to the version in Hunter and Nachtergaele's book, is the following: let Y be a subspace of X and ψ a linear functional on Y such that it is dominated by a sub-linear functional p, i.e., $\psi(x) \leq p(x) \forall x \in Y$. Then ψ can be extended to a functional Ψ on X such that $\Psi(x) \leq p(x) \forall x \in X$.
Show this implies the book's version of the theorem: if ψ is a bounded linear functional on Y, then it can be extended to a bounded linear functional Ψ on X such that $\|\Psi\| = \|\psi\|$.

(c) Prove the *hyperplane separation lemma* (a variant of this is known as the *geometric Hahn-Banach theorem*): let $C \subset X$ be a convex set with non-empty interior, and $d \in X \setminus C$. Prove there exists a separating hyperplane between C and d, i.e., there exists a linear function ψ on X such that $\psi(c) \leq \psi(d) \forall c \in C$.
Hint: Define an appropriate one-dimensional subspace and use part (a) of the problem as well as the stronger version of the Hahn-Banach theorem.