Applied Analysis Preliminary Exam
10.00am–1.00pm, August 20, 2013

Problem 1: Show that the non-linear integral equation:

$$v(x) = \cos^2(x) + \int_0^x e^{-v^2(s)} ds, \qquad x \in [0, \infty)$$

has a solution in $C^1([0,\infty),\mathbb{R})$.

Problem 2: Calculate the limit. **Justify** your answer.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\pi\sqrt{\frac{k}{n}}\right)\frac{1}{\sqrt{kn}}.$$

Problem 3: Given a self-adjoint compact operator $A : \ell^2 \longrightarrow \ell^2$, we define, for $\lambda \in \mathbb{R}$,

$$E_\lambda = \overline{\mathrm{Span}\{v \in \ell^2 \mid Av = \mu v \text{ for some } \mu \le \lambda\}}$$

and let

$$E^\lambda = E_\lambda^\perp$$

denote the orthogonal complement of E_λ.

(a) Show that E^1 is finite dimensional and A maps it to itself.

(b) In general, for what kind of value λ can you guarantee that:
  (1) E_λ is finite dimensional
  (2) E_λ is infinite dimensional
  (3) E^λ is finite dimensional
  (4) E^λ is infinite dimensional

Problem 4: Let H be a Hilbert space with an orthonormal basis $(\varphi_j)_{j=1}^\infty$. Suppose further that $(\lambda_j)_{j=1}^\infty$ is a sequence of non-negative real numbers such that $\lambda_j \to \infty$ as $j \to \infty$. Define for any finite positive integer n, the operator $A_n(t) \in \mathcal{B}(H)$ via

$$A_n(t)\, u = \sum_{j=1}^{n} e^{-\lambda_j t} \, (\varphi_j,\, u)\, \varphi_j.$$

(a) Show that for any $t \in [0,\infty)$, the sequence $(A_n(t))_{n=1}^\infty$ converges to some operator $A(t) \in \mathcal{B}(H)$. Specify the mode of convergence.

(b) Let $A(t)$ denote the limit operator defined in (a). Does $A(t)$ converge in $\mathcal{B}(H)$ as $t \searrow 0$? If yes, specify the mode.

Problem 5: Suppose that v is a real-valued function on $\mathbb{R}$ such that

$$\int_1^\infty |v(x)|^3 \, dx < \infty.$$

Define for $u \in C_{\mathrm{c}}(\mathbb{R})$ the functional

$$\varphi(u) = \int_1^\infty u(x)\, v(x)\, dx.$$

(a) For which $r \in [1,\infty)$ can you say for sure that φ can be extended to a continuous linear functional on $L^r(\mathbb{R})$?

(b) Suppose that in addition to $\int_1^\infty |v(x)|^3 \, dx < \infty$, you also know that v is a bounded function. Then for which $r \in [1,\infty)$ can you say for sure that φ can be extended to a continuous linear functional on $L^r(\mathbb{R})$?