Applied Analysis Preliminary Exam
10.00AM–1.00PM, AUGUST 21, 2012

INSTRUCTIONS. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your name on your exam. Each problem is worth 20 points.

Problem 1. Show that the initial value problem:

$$v'(x) = \frac{1}{4}\sin(x + v(x)^2), \qquad v(0) = \frac{1}{4},$$

has a unique solution in $C^2([0,1],\mathbb{R})$.

Problem 2. Define the right and left shift operators S and T on $\ell^2(\mathbb{N})$ by

$$S((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, \ldots), \quad \text{and} \quad T((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots).$$

You may use the relation $S^* = T$ without proving it. Prove the following:

(a) The point spectrum of T is the open disc $D = \{z \in \mathbb{C} : |z| < 1\}$ disc and S has no point spectrum.

(b) Every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ belongs to the continuous spectrum of S.

(c) The open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is contained in the residual spectrum of S.

(d) The continuous spectrum of S is the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$.

Problem 3. Let α be a complex number, and consider the operator

$$[Au](x) = \alpha\, u(x) + \arctan(x)\, u(x),$$

acting on the Hilbert space $H = L^2(\mathbb{R})$.

(a) What is the norm of A?

(b) For which values of α is A self-adjoint?

(c) For which values of α is A one-to-one?

(d) For which values of α is the range of A closed?

Problem 4. Let f be a non-negative, integrable function such that $\int_{\mathbb{R}} f < \infty$. Set for $t \geq 0$

$$g(t) = \int_{\mathbb{R}} e^{-tx^4\left(\sin\frac{1}{x^2+1}\right)} f(x)dx.$$

Show that:

(a) g is continuous on $[0, \infty)$.

(b) g is right differentiable at $t = 0$ if and only if $\int_{\mathbb{R}} x^2 f(x)dx$ is finite.

Problem 5. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{-tn}}{n}$ converges uniformly to a continuous function $f(t)$ on $[0, \infty)$. Show that $f(t) = -\int_t^\infty \frac{ds}{1+e^s}$. (Hint: compute $f'(t)$ term by term.)

Solution sketches:

Problem 1: Integrating the differential equation, we get

$$v(x) = \frac{1}{4} + \frac{1}{4}\int_0^x \sin(s + v^2(s))ds. \tag{1}$$

Let $||\psi||_{\mathrm{u}} = \sup_{t\in[0,1]} |u(t)|$ denote the uniform norm, and define the set

$$X = \{\phi \in C[0,1] \, : \, \phi(0) = \frac{1}{4} \text{ and } ||\phi||_{\mathrm{u}} \leq 1\}.$$

The set X combined with the uniform norm is a metric space. Now define the operator

$$[T\phi](x) = \frac{1}{4} + \frac{1}{4}\int_0^x \sin(s + (\phi(s))^2)ds,$$

the IVP can then be written as a fixed point problem $Tv = v$.

First observe that if $\phi \in X$, then $T\phi \in X$ as well. To verify this, observe that $[T\phi](0) = 1/4$, that $T\phi$ is continuous, and that $|[T\phi](x)| \leq \frac{1}{4} + \frac{1}{4}\int_0^x ds \leq 1/2$.

Next observe that T is a contraction on X. Indeed, if $\phi, \psi \in X$, then

$$\begin{aligned}||T\phi - T\psi||_{\mathrm{u}} &= \sup_x \frac{1}{4}\int_0^x \left|\sin(s + (\phi(s))^2) - \sin(s + (\psi(s))^2)\right| ds \leq \sup_x \frac{1}{4}\int_0^x \left|(\phi(s))^2 - (\psi(s))^2\right| ds \\ &\leq \sup_x \frac{1}{4}\int_0^x (|\phi(s)| + |\psi(s)|)\,|\phi(s) - \psi(s)|\, ds \leq \frac{1}{4}(||\phi||_{\mathrm{u}} + ||\psi||_{\mathrm{u}})\,||\phi - \psi||_{\mathrm{u}} \leq \frac{1}{2}||\phi - \psi||_{\mathrm{u}}.\end{aligned}$$

The contraction mapping theorem now asserts the existence of a unique $v \in X$ that solves $Tv = v$.

It remains to verify the claims on differentiability. Since v is continuous, (1) directly implies that v is C^1. Then the equation $v'(x) = (1/4)\sin(x + (v(x))^2)$ implies that v is C^2.

Problem 2:

(a) Consider the equation $Tx = \lambda x$. It has the only solution $x_n = \lambda^{n-1}x_1$. We see that $x \in \ell^2$ iff $|\lambda| < 1$. Next consider the equation $Sx = \lambda x$. If $\lambda = 0$, then clearly $x = 0$ so this is not an eigenvalue. If $\lambda \neq 0$, then the relation $0 = \lambda x_1$ implies that $x_1 = 0$, the relation $x_1 = \lambda x_2$ implies that $x_2 = 0$, etc.

(b) Set $D = \{z \in \mathbb{C} : \, |z| < 1\}$. We proved in (a) that $D \subset \sigma(T)$. Since $D = \overline{D}$ and $S = T^*$, this also shows that $D \subset \sigma(S)$. Since the spectrum is a closed set, we know that if $|\lambda| = 1$, then $\lambda \in \sigma(S)$. We showed in (a) that S does not have a point spectrum, so λ either belongs to the continuum or the residual spectrum. Now if $\lambda \in \sigma_{\mathrm{r}}(S)$, then $\overline{\lambda} \in \sigma_{\mathrm{p}}(T)$ since $S = T^*$, but this is impossible since we proved in (a) that the point spectrum of T equals the *open* unit disc. Therefore, $\lambda \in \sigma_{\mathrm{c}}(S)$.

(c) Let $\lambda \in D$. Then

$$\overline{\mathrm{ran}(S - \lambda I)} = \left(\ker(S^* - \bar{\lambda}I)\right)^{\perp} = \left(\ker(T - \bar{\lambda}I)\right)^{\perp}.$$

We proved in (a) that $\bar{\lambda}$ is an eigenvalue of T so $S - \lambda I$ is not dense in H. Since we also proved in (a) that λ is not an eigenvalue, it follows that $\lambda \in \sigma_{\mathrm{r}}(S)$.

(d) We showed in (b) that if $|\lambda| = 1$, then $\lambda \in \sigma_{\mathrm{c}}(S)$, and in (c) that if $|\lambda| < 1$, then $\lambda \in \sigma_{\mathrm{r}}(S)$. Since $||S|| = 1$, it follows that the spectrum of S is contained in the closed unit disc. Therefore $\sigma_{\mathrm{c}}(S) = C$.

Problem 3: Let I denote the line in the complex plane $I = \{z \in \mathbb{C} : \text{Im}(z) = 0 \; \text{Re}(z) \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$.

(a) Set $\alpha = \beta + i\gamma$ where β and γ are real. Set $C = \sup_{z \in I} |\alpha + z| = \sqrt{(\frac{\pi}{2} + |\beta|)^2 + \gamma^2}$. Since $|[Au](x)| \leq C|u(x)|$ for all x, we get $||A|| \leq C$. For the converse, suppose that $\beta \geq 0$ (the proof for $\beta < 0$ is analogous). Set $u_n = \chi_{[n,n+1]}$. Then $||u_n|| = 1$ and

$$||Au_n||^2 = \int_n^{n+1} |(\alpha + \arctan(x))|^2 \, dx = \int_n^{n+1} \big((\beta + \arctan(x))^2 + \gamma^2\big) \, dx \geq (\beta + \arctan(n))^2 + \gamma^2 \to C.$$

(b) We have

$$(Au, v) = \bar{\alpha} \int_{\mathbb{R}} \overline{u(x)} \, v(x) \, dx + \int_{\mathbb{R}} \arctan(x) \, \overline{u(x)} \, v(x) \, dx \tag{2}$$

$$(u, Av) = \alpha \int_{\mathbb{R}} \overline{u(x)} \, v(x) \, dx + \int_{\mathbb{R}} \arctan(x) \, \overline{u(x)} \, v(x) \, dx. \tag{3}$$

We see that A is self-adjoint if and only if α is real.

(c) Suppose that $Au = 0$. Then $(\alpha + \arctan(x)) \, u(x) = 0$ almost everywhere. This can happen only if $u = 0$. It follows that A is one-to-one for all α.

(d) If $\alpha \notin I$, then set $\delta = \min_{z \in I} |\alpha - z| = \text{dist}(I, \alpha)$. Since I is closed, $\delta > 0$. Clearly $||Au|| \geq \delta ||u||$, so A has closed range. To prove the converse, we will use that since A is one-to-one for all α, it has closed range if and only if it has a continuous inverse. Suppose first that $\alpha \in (-\pi/2, \pi/2)$. Set $I_n = (\tan(\alpha) - 1/n, \tan(\alpha) + 1/n)$ and $u_n = \chi_{I_n}$. Then $\lim_{n \to 0} ||Au_n||/||u_n|| = 0$, so A does not have a bounded inverse. If $\alpha = \pm\pi$, then use $u_n = \chi_{\pm[n,n+1]}$ to show that A is not coercive.

Problem 4: Since $g(t) = \int_R e^{-tx^4 \sin \frac{1}{1+x^2}} f(x)dx$ with $t \geq 0$ and $f \geq 0$, we get $h(t,x) = e^{-tx^4 \sin \frac{1}{1+x^2}} f(x) \leq f$, $\lim_{t \to t_0} h(t,x) = h(t_0, x)$. (i) By LBCT $\lim_{t \to t_0} g(t) = g(t_0)$ and $g(t)$ is continuous on $[0, \infty)$; (ii) If $x^2 f(x) \in L^1(R)$, then $|\frac{h(t+\delta,x) - h(t,x)}{\delta}| \leq x^4 \sin \frac{1}{1+x^2} f(x) \leq x^2 f(x)$ for $t \geq 0$, $t + \delta \geq 0$. We need to show $\frac{h(t+\delta,x)-h(t,x)}{\delta} \to x^4 \sin \frac{1}{1+x^2} e^{-tx^4 \sin \frac{1}{1+x^2}} f(x)$. By applying LBCT to the above equation, we get $g'(t) = \int_R x^4 \sin \frac{1}{1+x^2} e^{-tx^4 \sin \frac{1}{1+x^2}} f(x)dx$. (iii) If $g'(0^+)$ exists, then from Fatou's lemma, we get

$$\begin{aligned} -g(0^+) &= \lim_{t \to 0^+} \int_R \frac{f(x) - h(t,x)}{t} dx \geq \int_R \lim_{t \to 0^+} \frac{f(x) - h(t,x)}{t} dx \\ &= \int_R x^4 \sin \frac{1}{1+x^2} f(x) \geq \frac{2}{\pi} \int_R \frac{x^4}{1+x^2} f(x) dx. \end{aligned}$$

Hence, $\int_R x^2 f(x)dx \leq \int_R (\frac{x^4}{1+x^2} + 1) f(x)dx < \infty$.

Problem 5: Set

$$f_N(t) = \sum_{n=1}^{N} \frac{(-1)^n \, e^{-tn}}{n}.$$

To prove uniform convergence, we observe that for each fixed t, the sum passes the alternating sequence test, and therefore converges to some finite value which we call $f(t)$. Moreover, $|f_N(t) - f(t)| \leq \left| \frac{(-1)^{N+1} e^{-t(N+1)}}{N+1} \right| \leq \frac{1}{N+1}$. The convergence is therefore uniform. Since each f_N is continuous, it follows that f is continuous as well.

To prove the statement about the sum, we differentiate f_N to find

$$f_N'(t) = -\sum_{n=1}^{N} (-1)^n \, e^{-tn} = -\sum_{n=1}^{N} (-e^{-t})^n = -\frac{(-e^{-t}) - (-e^{-t})^{N+1}}{1 - (-e^{-t})} = \frac{1}{e^t + 1} + \frac{(-1)^{N+1} e^{-tN}}{e^t + 1}.$$

Since $\lim_{t \to \infty} f_N(t) = 0$, we have

$$f_N(t) = -\int_t^\infty f_N'(s) \, ds = -\int_t^\infty \frac{1}{e^s + 1} \, ds + (-1)^N \int_t^\infty \frac{e^{-sN}}{e^s + 1} \, ds.$$

The absolute value of the integrand in the second term is bounded by the L^1 function $g(t) = (e^t + 1)^{-1}$. We can therefore invoke dominated convergence as $N \to \infty$ to establish that the second term converges to zero.