Applied Analysis: Preliminary Exam

Department of Applied Mathematics, University of Colorado at Boulder

10.00am – 1.00pm, August 17, 2010

Problem 1: Set $I = [-1,\, 1]$ and define for $u \in C^2(I)$ the operator L via

$$[L\,u](x) = -\frac{d}{dx}(1-x^2)\frac{d}{dx}\,u(x).$$

Set

$$\Omega = \{L\,u :\ u \in C^2(I)\}.$$

(a) Find a function $u \in C^2(I)$ such that $[L\,u](x) = x$.

(b) Show that $\Omega \subseteq L^2(I)$.

(c) For a function $f \in \Omega$, give an explicit formula for a function $u \in C^2(I)$ such that $L\,u = f$. (Your formula may involve unevaluated integrals, and/or sums of unevaluated integrals.)

(d) Describe the topological closure $\overline{\Omega}$ of Ω in $L^2(I)$. (For any $f \in \overline{\Omega}$, the equation $L\,u = f$ has a solution $u \in L^2(I)$ when the differential operator L is defined in a "weak" sense.)

Hint for Problem 1: Define for $n = 0,\, 1,\, 2,\, 3,\, \dots$ the functions Q_n via

$$Q_n(x) = \sqrt{\frac{2n+1}{2}}\,\frac{1}{2^n\,n!}\left(\frac{d}{dx}\right)^n \left(x^2-1\right)^n. \tag{1}$$

You may use that

$$L\,Q_n = n\,(n+1)\,Q_n, \tag{2}$$

and that $\{Q_n\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2(I)$.

Problem 2: Specify which of the following statements are true. No justification necessary.

(a) The set of even functions is dense in $L^2([-1,\, 1])$.

(b) The set of polynomials is dense in $L^2([-1,\, 1])$.

(c) The set of simple functions is dense in $L^2(\mathbb{R})$. (Recall that a *simple function* is a function of the form $u = \sum_{j=1}^{J} c_j\,\chi_{\Omega_j}$ where J is a finite integer, c_j is a scalar, and Ω_j is a measurable subset of $\mathbb{R}$.)

(d) The set of bounded continuous functions is dense in $L^\infty(\mathbb{R})$.

(e) The set $C^1([-1,\, 1])$ is dense in $C([-1,\, 1])$.

(f) The space $L^p(\mathbb{R})$ is separable for all p such that $1 \le p < \infty$.

(g) The space $\ell^p(\mathbb{N})$ is separable for all p such that $1 \le p \le \infty$.

(h) The space $C([-1,\, 1])$ is separable.

Problem 3: Let $V = \ell^1(\mathbb{N})$ denote the set of all real-valued sequences $\mathbf{x} = (x_1,\, x_2,\, x_3,\, \dots)$ such that

$$||\mathbf{x}|| = \sum_{j=1}^{\infty} |x_j| < \infty. \tag{3}$$

For $\mathbf{x} \in V$, we define the function

$$n(\mathbf{x}) = \sup_{\Omega \subseteq \mathbb{N}} \left| \sum_{j \in \Omega} x_j \right|. \tag{4}$$

(a) Prove that the function n defined via (4) is a norm on V.

(b) Prove that $||\cdot||$ and n are equivalent norms on V.

(c) Let W denote the set of all $\mathbf{x} = (x_1,\, x_2,\, x_3,\, \dots) \in V$ such that $\sum_{j=1}^{\infty} x_j = 0$.
Is W a linear subspace of $\ell^1(\mathbb{N})$? Is W a compact set in $\ell^1(\mathbb{N})$?

Problem 4: Set $I = [0,\, 1]$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on I that converges point-wise to some function f. Let $k : I^2 \to I$ be a continuous function. Prove that if

$$f_{n+1}(x) = \int_0^x k(f_n(y), y)\, dy, \qquad n = 1,\, 2,\, 3,\, \dots,$$

then f is continuous. (You may assume that $0 \leq f_1(x) \leq 1$ for $x \in I$.)

Problem 5: Let H be a Hilbert space, and let A be a bounded linear operator on H such that $A^3 = 0$.

(a) Prove that for any complex number z, the operator $B = I - z\,A$ is invertible.

(b) What can be said about the spectrum of A?

(c) Give an example of a Hilbert space H and a non-zero operator $A \in \mathcal{B}(H)$ such that $A^3 = 0$. Is it possible to find such an example that also satisfies $A = A^*$? Motivate.