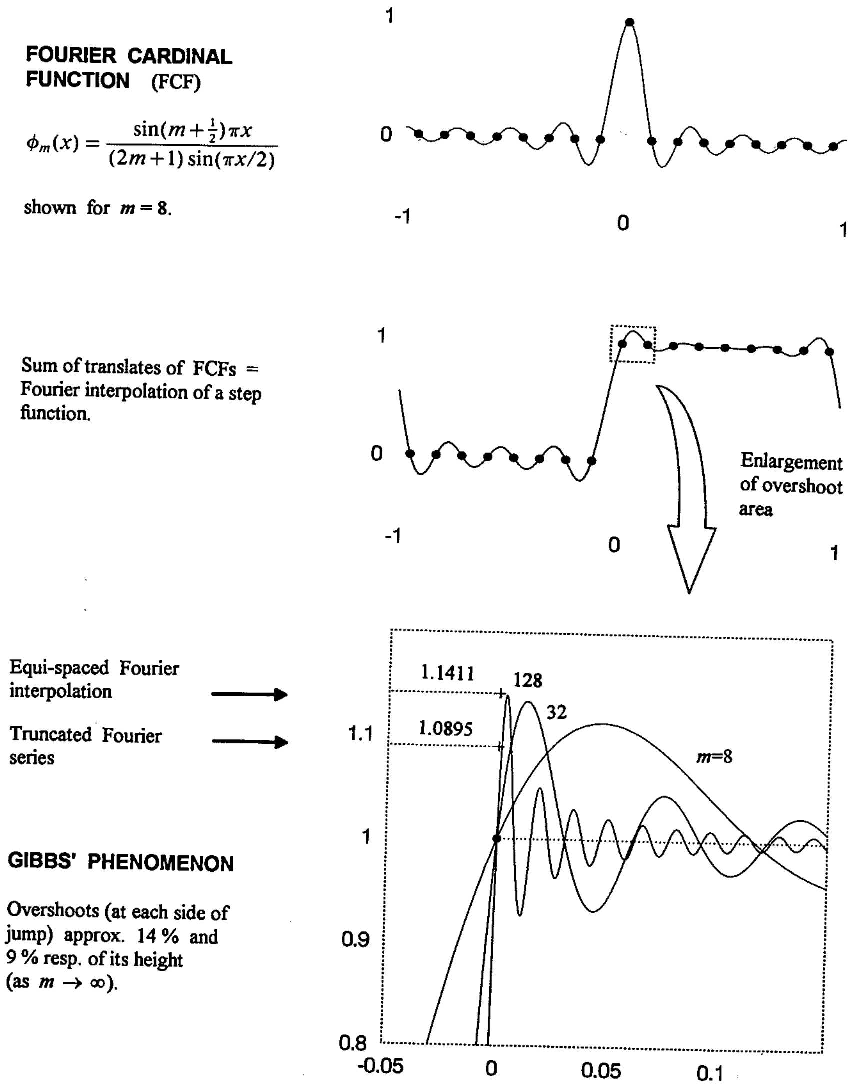

Figure 2.3-1. Fourier cardinal functions and Gibbs' phenomenon.

where the differentiation matrix D has the elements (making use of equation (2.3-3)):

$$D_{i,j} = \frac{d}{dx}\phi_m(x)\bigg|_{x=x_{i-j}} = \begin{cases} \dfrac{\pi(-1)^{i-j}}{2\sin\pi((i-j)/N)} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases} \qquad (2.3\text{-}4)$$

2.4. Gibbs' phenomenon

The overshoot shown in the lower portion of Figure 2.3-1 arises whenever a discontinuous function is expanded in or interpolated with smooth functions.

What we now call Gibbs' phenomenon was first noted by Wilbraham (1848). Unaware of this, Michelson and Stratten (1898) found traces of these overshoots in the output plots from a mechanical Fourier analyzer they had constructed.

Michelson and Stratten's analyzer is described in some detail under the entry "Calculating machines" in the 1910 (11th) edition of The Encyclopedia Britannica. Some hardware is preserved (but not normally on display) at the Smithsonian in Washington, DC. The analyzer was a refinement (and extension to about 80 modes) of an earlier version invented by Lord Kelvin for the calculation of tides. Kelvin's device was so well suited for its task that it remained in use 20 years into the era of electronic computers.

This observation by Michelson (who is probably best known for this ether experiment with Morley) prompted him to write a letter to *Nature* inquiring about the convergence properties of a Fourier series for a discontinuous function. In reply, J. Gibbs (an eminent chemist) provided first a flawed and then a satisfactory answer. Some historical notes on Gibbs' phenomenon can be found in Hewitt and Hewitt (1979).

Two different variations of Gibbs' phenomenon arise in spectral methods. The overshoots on a jump of height 1 become as follows.

Equi-spaced Fourier interpolation. The notation will be simpler if we first transform to an infinite interval. Following the line of reasoning indicated in Figure 2.3-1 and noting equation (2.3-2), we have:

$$G_I = \max_{0<\xi<1}\left\{\frac{\sin\pi\xi}{\pi\xi} + \frac{\sin\pi(\xi-1)}{\pi(\xi-1)} + \frac{\sin\pi(\xi-2)}{\pi(\xi-2)} + \cdots\right\} - 1$$
$$= \max_{0<\xi<1}\left\{\frac{\sin\pi\xi}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\xi-k}\right\} - 1 \approx 0.1411.$$

Truncated Fourier expansion. We consider again a piecewise constant function with a unit jump at the origin:

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < \pi, \\ -\frac{1}{2} & \text{if } -\pi < x < 0. \end{cases}$$

The Fourier series (of a 2π-periodic extension) of this function is

$$f(x)=\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}.$$

The derivative of its truncated sum

$$f_N(x)=\frac{2}{\pi}\sum_{k=0}^{N}\frac{\sin(2k+1)x}{2k+1}$$

is

$$f_N'(x)=\frac{2}{\pi}\sum_{k=0}^{N}\cos(2k+1)x=\frac{\sin 2(N+1)x}{\pi\sin x};$$

that is, the extrema of $f_N(x)$ for $x>0$ occur at $x_{N,j}=\pi j/2(N+1)$, $j=1,2,\ldots$. At these points,

$$f_N(x_{N,j})=\frac{1}{\pi(N+1)}\sum_{k=0}^{N}\frac{\sin((2k+1)\pi j/2(N+1))}{(2k+1)/2(N+1)}.$$

The sum is a discrete approximation to the integral

$$\frac{1}{\pi}\int_0^1\frac{\sin j\pi t}{t}\,dt=\frac{1}{\pi}\int_0^{j\pi}\frac{\sin t}{t}\,dt.$$

The maximum value is taken for $j=1$, yielding $(1/\pi)\int_0^\pi((\sin t)/t)\,dt\approx 0.5895$. Thus, as $N\to\infty$, the overshoot of a unit-height jump approaches $G_T\approx 0.0895$.

Gibbs' phenomenon for Chebyshev (Jacobi) expansions is essentially the same as in the Fourier case when the irregularities are located inside $[-1,1]$. However, the interpolation overshoot from a unit jump at $x=\pm1$ is larger:

$$G_{I,\pm1}=\max_{1\le\xi\le2}\left\{-\frac{\sin\pi\xi}{\pi\xi}\right\}\approx 0.2172.$$

Gibbs' phenomenon (the $O(1)$ error next to a discontinuity) is the most notable instance of how an irregularity of a piecewise smooth function can affect the convergence of both interpolants and truncated series expansions; see Table 2.4-1. The decay rates of Fourier expansion coefficients are the same as the order of the maximum norm of errors away from irregularities.

For continuous but not piecewise differentiable functions, a Fourier series can do such strange things as diverging to infinity at some point(s) (in spite of each truncation giving the best possible least-squares approximation to the function when using up to that number of terms!). Such subtleties have no numerical consequences – Table 2.4-1 is a good guide for all situations of numerical relevance.

Table 2.4-1. *Order of max-norm errors caused by irregularities of a function*

Function	Max-norm of errors (order): Near irregularity	Max-norm of errors (order): Away from irregularity
f discontinuous	1	$1/N$
f' discontinuous	$1/N$	$1/N^2$
f'' discontinuous	$1/N^2$	$1/N^3$
$\vdots$	$\vdots$	$\vdots$
f analytic (periodic)	e^{-cN}, $c>0$	